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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

Asymmetric and Axisymmetric Vibrations of Finite Transversely Isotropic Circular Cylinders[1]

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Abstract—This paper presents an approach for obtaining the exact frequency equations of axisymmetric and asymmetric free vibrations of transversely isotropic circular cylinders. The solution method is based on the three dimensional theory of linear elasticity and uses potential functions. Using this approach, the frequency spectra and vibration mode shapes are plotted for a number of transversely isotropic cylinders. The proposed approach introduces a number of merits compared to earlier approximate and exact solution methods. First, unlike numerically complicated series methods that provide approximate solutions, the proposed approach is exact. Second, combination of scalar functions employed for representing the displacement field is consistent with the physics of the problem. One scalar potential function has been considered for each component of the wave field inside the elastic cylinder. As a result, the solution is systematically divided into coupled and decoupled equations. In addition, by using this approach, there is no need to guess the final of the solution a priori. These merits make the proposed approach suitable for other vibration problems of anisotropic materials.

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1. INTRODUCTION

A number of theories for determination of natural frequencies of cylinders have been developed and used over the years [1–4]. For a finite-length vibrating cylinder, an exact mathematical solution cannot be easily obtained by the three-dimensional linear elasticity theory. Therefore, several approximate solutions have been proposed for estimating the natural frequencies and mode shapes of isotropic cylinders [3–11]. When the cylinder is anisotropic, the solution of the free vibration problem becomes significantly more complex. Lusher and Hardy [1] used the approximate series approach of Morse [13] to study the axisymmetric modes of finite-length transversely isotopic cylinders and presented experimental results for sapphire rods. Heyliger [12] applied the Ritz method for estimating the natural frequencies of axisymmetric free vibrations of finite transversely isotopic cylinders. Using numerical and analytical approaches, Grigorenko [14] studied the natural vibrations of anisotropic solid and hollow cylinders of finite-length with different end conditions. He also investigated the dependence of the dynamic characteristics of the cylinder on its geometrical and mechanical parameters. The most complete study of vibrations of finite-length transversely isotopic cylinders is due to Chau [2]. He obtained the axisymmetric and asymmetric frequency equations of a finite-length transversely isotopic cylinder by using a potential function approach. Although each of the methods discussed above has considered a particular aspect of the vibrations of finite-length isotropic and transversely isotropic cylinders, each method has some limitations as follows.

Due to the nature of the series solution used by Lusher and Hardy [1], their method can provide answers as close to exact as required for a transversely isotropic cylinder, but it is numerically complicated. Moreover, their method leads to complex terms for displacement components and makes the solution of the frequency equation complicated at high frequencies. In the Rayleigh–Ritz [8] or Ritz [11] approximate methods, the weak forms of the governing equations are solved by using approximate functions. Consequently, the frequency spectra obtained from this method are approximate. In order to match the approximate spectra with the exact spectra, several adjustment factors are introduced into the solution. These additional steps increase the complexity of the method, and may lead to errors in the final solution.

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Chau's potential function method for transversely isotropic finite-length cylinders [2] is the only exact solution available. However, in this approach, the form of the final solution is assumed a priori, and there is no physical explanation for the assumed solution.

Moreover, except for the finite element analysis of Gladwell and Vijay [7], the Ritz analysis of Heyliger and Jilani [12] for isotropic materials, and the potential functions method of Chau [2] for transversely isotropic cylinders, all other methods are restricted to axisymmetric modes of vibration.

In this paper, an approach is proposed for obtaining the axisymmetric and asymmetric frequency equations of finite-length transversely isotropic cylinders. This approach was initially developed by the authors for studying the propagation/scattering of elastic waves in/from infinite transversely isotropic cylinders [15, 17]. In this paper, this approach is extended to finite-length transversely isotropic circular cylinders. The proposed approach employs potential functions to provide exact frequency equations through a systematic method using the exact 3D elasticity equations. The form of the potential functions used is consistent with the physics of the problem which makes the proposed approach distinct from other solution techniques and introduces a number of merits for it. Consistency with the physics of the problem, and relative simplicity due to a systematic solution are two of these merits. In addition, this method can easily be extended to more complex problems.

2. PROBLEM FORMULATION

We consider a finite-length transversely isotopic circular cylinder. The periphery of the cylinder is traction-free and its ends are constrained by frictionless rigid walls. A cylindrical coordinate system, (r, θ, z), is chosen with the z-direction coincident with the axis of the cylinder, see Fig. 1.

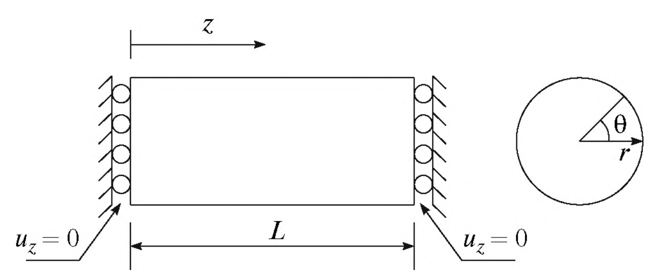

Fig. 1. The coordinate system used in the derivation of equations.

Wave equations are obtained by combining the constitutive equations and equations of motion for a transversely isotopic material [15],

$$
\begin{aligned}
&c_{44}\left(\frac{\partial^2 u_z}{\partial z \partial r} + \frac{\partial^2 u_r}{\partial z^2}\right) + c_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{3}{2r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} + \frac{1}{2r}\frac{\partial^2 u_\theta}{\partial r \partial \theta}\right) \\
&+ \frac{1}{2r}c_{12}\left[\frac{\partial}{\partial \theta}\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\theta}{r}\right)\right] + c_{13}\frac{\partial^2 u_z}{\partial z \partial r} = \rho_c \frac{\partial^2 u_r}{\partial t^2},
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
&c_{11}\left(\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{1}{2r}\frac{\partial u_\theta}{\partial r} + \frac{1}{2}\frac{\partial^2 u_\theta}{\partial r^2} + \frac{3}{2r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 u_r}{\partial r \partial \theta} - \frac{1}{2}\frac{u_\theta}{r^2}\right) \\
&+ c_{44}\left[\frac{\partial}{\partial z}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)\right] + \frac{c_{13}}{r}\frac{\partial^2 u_z}{\partial z \partial \theta} = \rho_c \frac{\partial^2 u_\theta}{\partial t^2},
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&c_{44}\left(\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial z \partial \theta} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_r}{\partial z \partial r}\right) \\
&+ c_{13}\left[\frac{\partial}{\partial z}\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_r}{\partial r}\right)\right] + c_{33}\frac{\partial^2 u_z}{\partial z^2} = \rho_c \frac{\partial^2 u_z}{\partial t^2},
\end{aligned} \tag{3}
$$

where ρ_c is the cylinder density, t is time, and u_r, u_θ, and u_z are the displacement components in the r, θ, and z directions, respectively. The five independent elastic constants of a transversely isotopic material are denoted by c_{11}, c_{12}, c_{13}, c_{33}, and c_{44}.

From a physical point of view and in the most general form, the wave field inside an elastic medium can be decomposed into three independent components, i.e., one compression (P) and two shear (SH and SV) waves [16]. This physical picture is mathematically addressed by allocating one scalar potential function to each of these three wave components. However, according to the second theorem of Helmholtz, these three independent components should be mutually orthogonal [16]. That is why the displacement field inside a circular cylinder is written in terms of three scalar potential functions ϕ, χ, and ψ in an orthogonal form as follows [17],

$$
\mathbf{U} = \nabla\phi + \nabla \times (\chi \hat{e}_z) + a\nabla \times \nabla \times (\psi \hat{e}_z), \tag{4}
$$

where $\nabla\phi$ represents the compression wave and $\nabla \times (\chi \hat{e}_z)$ and $\nabla \times \nabla \times (\psi \hat{e}_z)$ correspond to SH and SV waves, respectively. Moreover, a is the radius of the cylinder and ∇ and $\nabla \times$ are the gradient, and curl operators, respectively. By substituting Eq. (4) into Eqs. (1)–(3), the wave equations are obtained in terms of the three potential functions,

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\left\{c_{11}\nabla^2\phi + (c_{13} + 2c_{44} - c_{11})\frac{\partial^2\phi}{\partial z^2} - \rho_c\frac{\partial^2\phi}{\partial t^2} + a\frac{\partial}{\partial z}\left[(c_{11} - c_{13} - c_{44})\nabla^2\psi + (c_{13} + 2c_{44} - c_{11})\frac{\partial^2\psi}{\partial z^2} - \rho_c\frac{\partial^2\psi}{\partial t^2}\right]\right\} = 0, \tag{5}$$

$$\frac{\partial}{\partial z}\left[(c_{13} + 2c_{44})\nabla^2\phi + (c_{33} - c_{13} - 2c_{44})\frac{\partial^2\phi}{\partial z^2} - \rho_c\frac{\partial^2\phi}{\partial t^2}\right] + a\left(\frac{\partial^2}{\partial z^2} - \nabla^2\right) \times\left[c_{44}\nabla^2\psi + (c_{33} - c_{13} - 2c_{44})\frac{\partial^2\psi}{\partial z^2} - \rho_c\frac{\partial^2\psi}{\partial t^2}\right] = 0, \tag{6}$$

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\left[\frac{(c_{11} - c_{12})}{2}\nabla^2\chi + \left(c_{44} - \frac{(c_{11} - c_{12})}{2}\right)\frac{\partial^2\chi}{\partial z^2} - \rho_c\frac{\partial^2\chi}{\partial t^2}\right] = 0. \tag{7}$$

The immediate consequence of representing the displacement field by three mutually orthogonal terms is that the three Eqs. (1)–(3), which are coupled, would turn into two coupled equations, i.e. Eqs. (5) and (6), and one decoupled equation, i.e. Eq. (7). This decoupling allows one to employ the method of separation of variables for solving the problem as a relative simple and straightforward method [17]. This property makes the proposed approach distinct from its predecessors. The forms of the solutions of these equations which are used in [17] correspond to propagating waves (for infinite cylinders) and are not suitable for the current problem which deals with a finite length cylinder. The solutions sought for the current problem should be in the form of standing waves which could form in a finite-length cylinder. Therefore, they should be considered as,

$$\phi(r, \theta, z, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(B_nJ_n(s_1r) + q_2C_nJ_n(s_2r)) \times\cos(n\theta)\cos(\eta z)e^{-i\omega t}, \tag{8}$$

$$\psi(r, \theta, z, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(q_1B_nJ_n(s_1r) + C_nJ_n(s_2r)) \times\cos(n\theta)\sin(\eta z)e^{-i\omega t}, \tag{9}$$

$$\chi(r, \theta, z, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}D_nJ_n(s_3r)\sin(n\theta)\cos(\eta z)e^{-i\omega t}, \tag{10}$$

where $\eta = m\pi/L$, $m = 1, 2, \ldots$, L is the length for the cylinder, m is the mode number in z direction, ω is the circular frequency, and J_n are first type Bessel functions of order n. Moreover, s_1, s_2, s_3, q_1 and q_2 are constants which depend on the elastic constants of the material as well as the frequency and η. The mode numbers $n = 0$ and $n = 1$ correspond to the axisymmetric and first asymmetric vibration modes, respectively. Full expressions of the above mentioned parameters can be found in Appendix A. Substituting Eqs. (8)–(10) into Eq. (4) gives exact displacement distribution of a vibrating circular cylinder for different modes, materials, and boundary conditions.

We consider the vibrations of a finite cylinder subjected to zero shear traction and zero axial displacement on end surfaces following [1]. Lusher and Hardy [1] found that their measured natural frequencies agree better with the theoretical predictions if the vibration modes are calculated based on small axial displacements on end surfaces. Accordingly, a zero displacement end condition was more appropriate on modeling their experimental configuration. Therefore, in our model, we also consider the following boundary conditions,

$$u_z = 0, \quad \sigma_{rz} = 0, \quad \sigma_{z\theta} = 0, \quad \text{at} \quad z = 0, L \tag{11}$$

(lateral boundary conditions)

$$\sigma_{rr} = 0, \quad \sigma_{rz} = 0, \quad \sigma_{r\theta} = 0, \quad \text{at} \quad r = a \tag{12}$$

(transverse boundary conditions).

The "transverse" boundary conditions do not appear in the frequency equation while the "lateral" boundary conditions enter into the frequency equation directly. The physical implication of this difference for a cylindrical geometry is that the transverse conditions control the "structural" pieces of the solution and determine which terms are allowed to contribute in the construction of the solution, see Eqs. (8)–(10). Moreover, lateral conditions determine what mixture of these pieces is required for constructing the frequency equation, i.e. Eq. (13).

Exact expanded expressions for the stress and displacement at any point can be derived in terms of potential functions. Substituting the potential functions which were introduced in Eqs. (8)–(10) into Eq. (12) results in the following system of linear algebraic equations,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = 0. \tag{13}$$

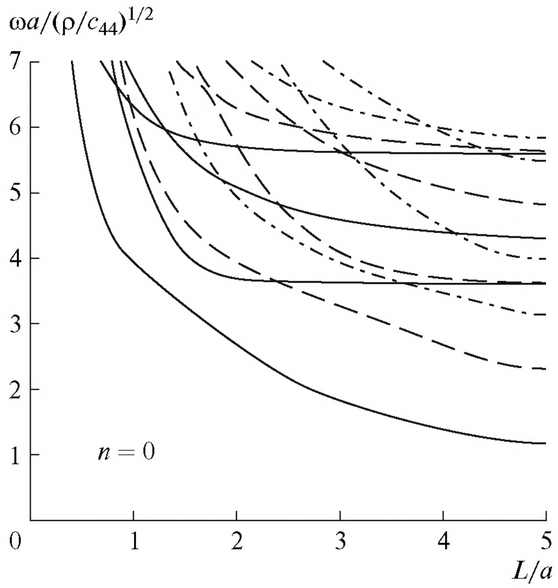

Fig. 2. Frequency spectrum of a sapphire rod, for $n = 0$ and $m = 1, 2, 3$ ($m = 1$: solid line, $m = 2$: dashed line, $m = 3$: broken line).

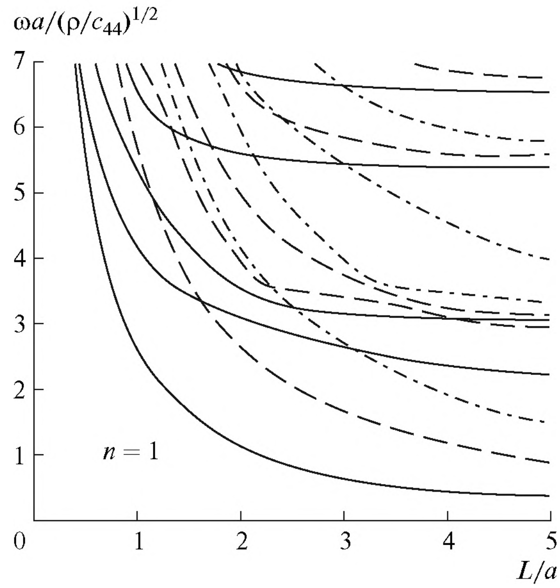

Fig. 3. Frequency spectrum of a sapphire rod, for $n = 1$ and $m = 1, 2, 3$ ($m = 1$: solid line, $m = 2$: dashed line, $m = 3$: broken line).

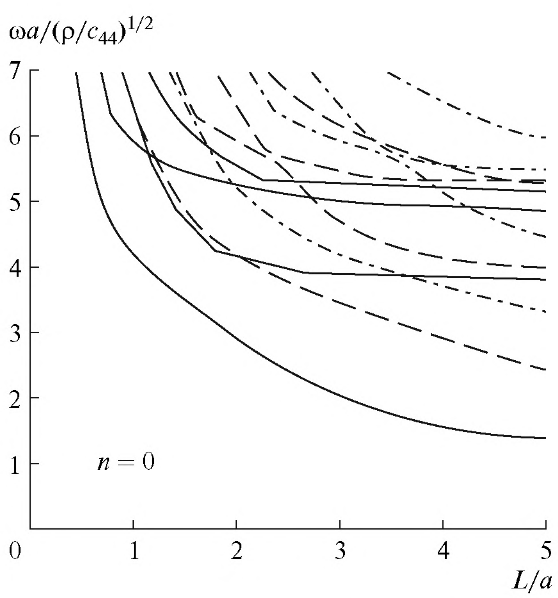

Fig. 4. Frequency spectrum of a cobalt rod for $n = 0$ and $m = 1, 2, 3$ ($m = 1$: solid line, $m = 2$: dashed line, $m = 3$: broken line).

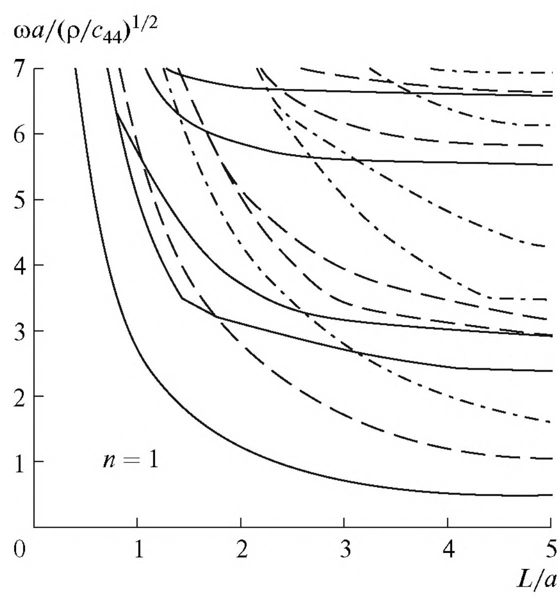

Fig. 5. Frequency spectrum of a cobalt rod for $n = 1$ and $m = 1, 2, 3$ ($m = 1$: solid line, $m = 2$: dashed line, $m = 3$: broken line).

The expressions for a_{ij} are given in Appendix B. The solution to Eq. (13) is nontrivial only if the determinant of the coefficient matrix vanishes, i.e.,

$$\det\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \qquad (14)$$

Equation (14) is the exact frequency equation for both asymmetric and axisymmetric modes. Reduction of Eq. (14) to the case of $n = 0$ gives the frequency equation for axisymmetric modes,

$$\det(a_{ij}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0. \qquad (15)$$

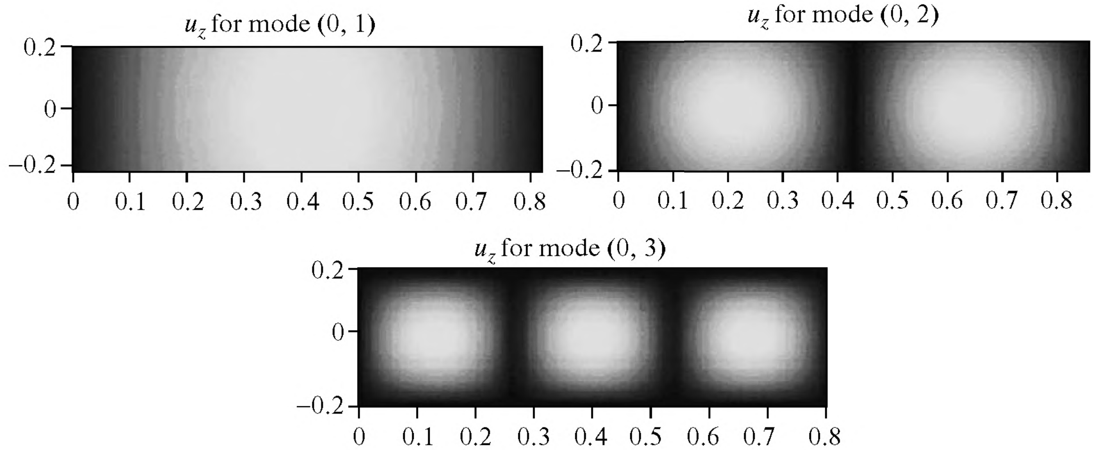

Fig. 6. Displacement field u_z inside a sapphire rod for modes (0, 1), (0, 2), (0, 3).

3. NUMERICAL RESULTS AND DISCUSSIONS

In traditional methods, the natural frequencies of a vibrating medium are found by iterative root-finding algorithms that extract the zeros of frequency equations. Since iterative root-finding algorithms are prone to numerical errors, an alternative method proposed in [18] is employed for solving the frequency equations. In this method, the three-dimensional representation of the frequency equation is used for obtaining the frequency spectra.

Figures 2–5 show the frequency spectra for sapphire rods with the length-to-radius ratio (L/a) from 0 to 5. Figures 2 and 3 are plotted for the same sapphire rod considered in [1] and [2]. These spectra are in excellent agreement with those given in [1] and [2] and confirm the validity of the approach developed in this paper. The frequency spectra for $n = 0$ (axisymmetric) and $n = 1$ (the first asymmetric) modes of a cobalt transversely isotropic cylinder are also shown in Figs. 4 and 5, respectively. The physical parameters of both sapphire and cobalt can be found in table.

Another effective method for examining the correctness of the solutions of a frequency spectrum is examining the mode shape at each point on the modal curve of the frequency spectrum. Each curve in the frequency spectrum corresponds to a specific mode which has its unique mode shape. Figure 6 shows the displacement field inside a sapphire rod with length-to-radius ratio of 4 for modes (0, 1), (0, 2), (0, 3). As shown in Fig. 6, the number of nodes and antinodes inside the cylinder is controlled by m in the z direction and they increase as mode number increases. The light and dark areas in Fig. 6 correspond to nodes and antinodes, respectively.

Physical parameters

Material	Stiffness × 10^{11}, N/m^2					Density, kg/m^3
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	
Sapphire	4.968	1.636	1.109	4.981	1.474	3986
Cobalt	2.95	1.59	1.11	3.35	0.71	8900

4. CONCLUSIONS

In this paper, a solution method was proposed for studying the vibrations of finite-length transversely isotropic cylinders. Exact axisymmetric and asymmetric vibrating modes were obtained based on the three-dimensional theory of elasticity. To verify the proposed approach, the frequency spectra of axisymmetric and asymmetric vibration modes of rods made from sapphire and cobalt were calculated. In addition, vibration mode shapes were plotted for the sapphire rod at its resonance frequencies. This approach produces numerical results which are in excellent agreement with published experimental and theoretical data and has a number of advantages compared to earlier approaches. Consistency with the physics of the problem, exactness, and relative simplicity are among these advantages that make this approach a useful tool to study the vibrations of transversely isotropic cylinders. Moreover, this physical consistency introduced a generic concept and accordingly a generic analytical approach that could be used for tackling other vibration problems of anisotropic structures.

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APPENDIX A

Substituting Eqs. (8)–(10) into the governing equations, Eqs. (5)–(7) yields an eigenvalue problem in which the eigenvalues are the velocities of vibrating modes and the eigenvectors are their corresponding polarization vectors, i.e.,

$$\begin{bmatrix} -s^2(c_{11}s^2-(\rho_c\omega^2-(c_{13}+2c_{44})k_z^2)) & -aik_zs^2((c_{11}-c_{13}-c_{44})s^2-(\rho_c\omega^2-(c_{44}k_z^2))) & 0 \\ ik_z((c_{13}+2c_{44})s^2-(\rho_c\omega^2-c_{33}k_z^2)) & as^2(c_{44}s^2-(\rho_c\omega^2-(c_{33}-c_{13}-c_{44})k_z^2)) & 0 \\ 0 & 0 & -s^2((c_{11}-c_{12})s^2-2(\rho_c\omega^2-c_{44}k_z^2)) \end{bmatrix}\begin{bmatrix} B_n \\ C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{(A.1)}$$

To have nontrivial solutions for Eq. (A.1), its determinant should be equal to zero which gives the following characteristic equation;

$$(s^2+k_z^2)(c_{44}c_{11}s^4-\xi s^2+\zeta)\times\left(\left(\frac{c_{11}-c_{12}}{2}\right)s^2-\rho_c\omega^2+c_{44}k_z^2\right) = 0, \quad \text{(A.2)}$$

where,

$$\xi = (c_{13}+c_{44})^2k_z^2+c_{11}(\rho_c\omega^2-c_{33}k_z^2) + c_{44}(\rho_c\omega^2-c_{44}k_z^2), \quad \text{(A.3)}$$

$$\zeta = (\rho_c\omega^2-c_{44}k_z^2)(\rho_c\omega^2-c_{33}k_z^2). \quad \text{(A.4)}$$

There are three meaningful solutions (eigenvalues), s_1, s_2 and s_3, for Eq. (A.2),

$$s_1^2 = \frac{\xi-\sqrt{\xi^2-4c_{11}c_{44}\zeta}}{2c_{11}c_{44}}, \quad \text{(A.5)}$$

$$s_2^2 = \frac{\xi+\sqrt{\xi^2-4c_{11}c_{44}\zeta}}{2c_{11}c_{44}}, \quad \text{(A.6)}$$

$$s_3^2 = \frac{2(\rho_c\omega^2-c_{44}k_z^2)}{c_{11}-c_{12}}. \quad \text{(A.7)}$$

The coefficients of q_1 and q_2 which couple the ϕ and ψ are as follows,

$$q_1 = -\frac{(c_{11}s_1^2-(\rho_c\omega^2-(c_{13}+2c_{44})k_z^2))}{aik_z((c_{11}-c_{13}-c_{44})s_1^2-(\rho_c\omega^2-(c_{44}k_z^2)))}, \quad \text{(A.8)}$$

$$q_2 = -\frac{aik_z((c_{11}-c_{13}-c_{44})s_2^2-(\rho_c\omega^2-(c_{44}k_z^2)))}{(c_{11}s_2^2-(\rho_c\omega^2-(c_{13}+2c_{44})k_z^2))}. \quad \text{(A.9)}$$

APPENDIX B

Elements of the matrices given in Eqs. (14) and (15) are as follows,

$$\begin{aligned} a_{11} &= [c_{11}+a\eta q_1(c_{11}-c_{13})] \\ &\times[(n^2-n-s_1^2a^2)J_n(s_1a)+(s_1a)J_{n+1}(s_1a)] \\ &+[c_{12}+a\eta q_1(c_{12}-c_{13})] \\ &\times[n^2J_n(s_1a)-(s_1a)J_{n+1}(s_1a)] \\ &+(-c_{13}\eta^2a^2-c_{12}n^2+an^2\eta q_1(c_{13}-c_{12}))J_n(s_1a), \end{aligned} \quad \text{(B.1)}$$

$$\begin{aligned} a_{12} &= [c_{11}q_2+\eta a(c_{11}-c_{13})] \\ &\times[(n^2-n-s_2^2a^2)J_n(s_2a)+(s_2a)J_{n+1}(s_2a)] \\ &+[c_{12}q_2+\eta a(c_{12}-c_{13})] \\ &\times[n^2J_n(s_2a)-(s_2a)J_{n+1}(s_2a)] \\ &+(-c_{13}q_2\eta^2a^2-c_{12}q_2n^2+n^2\eta a(c_{13}-c_{12}))J_n(s_2a), \end{aligned} \quad \text{(B.2)}$$

$$\begin{aligned} a_{13} &= n(c_{11}-c_{12}) \\ &\times[(n-1)J_n(s_3a)-(s_3a)J_{n+1}(s_3a)], \end{aligned} \quad \text{(B.3)}$$

$$\begin{aligned} a_{21} &= c_{44}[q_1(s_1^2a^2-\eta^2a^2)+2\eta a] \\ &\times[nJ_n(s_1a)-(s_1a)J_{n+1}(s_1a)], \end{aligned} \quad \text{(B.4)}$$

$$\begin{aligned} a_{22} &= c_{44}[(s_1^2a^2-k^2a^2)+2\eta aq_2] \\ &\times[nJ_n(s_2a)-(s_2a)J_{n+1}(s_2a)], \end{aligned} \quad \text{(B.5)}$$

$$a_{23} = c_{44}(-n\eta a)J_n(s_3a), \quad \text{(B.6)}$$

$$\begin{aligned} a_{31} &= n(c_{11}-c_{12})(1+\eta aq_1) \\ &\times[(1-n)J_n(s_1a)+(s_1a)J_{n+1}(s_1a)], \end{aligned} \quad \text{(B.7)}$$

$$\begin{aligned} a_{32} &= n(c_{11}-c_{12})(q_2+\eta a) \\ &\times[(1-n)J_n(s_2a)+(s_2a)J_{n+1}(s_2a)], \end{aligned} \quad \text{(B.8)}$$

$$\begin{aligned} a_{33} &= \left(\frac{c_{11}-c_{12}}{2}\right) \\ &\times\{[s_3^2a^2-2n(n-1)]J_n(s_3a)-2(s_3a)J_{n+1}(s_3a)\}. \end{aligned} \quad \text{(B.9)}$$

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