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Conditions for Forming Weakly Diverging Acoustic Bundles in Ocean Waveguides

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Abstract—Necessary and sufficient conditions for the dependence of the cycle length of the Brilloin waves or rays of geometrical acoustics on the ray parameter that is inversely proportional to their phase velocity are formulated. The formulated conditions determine the existence of weakly diverging acoustic bundles in vertically stratified oceanic waveguides, even with a point sound source.

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INTRODUCTION

The objective of this work consists in determining the conditions that are necessary and sufficient for weakly diverging ray bundles to be formed in vertically stratified oceanic waveguides with a point source of sound waves. Naturally, those questions have been widely discussed earlier [1–10]. However, more recent publications [11–13] that offer some contradictory conclusions made it mandatory to analyze the problem in more detail. Such an analysis is important in interpreting the laws of forming the spatial (in depth and horizontal distance, [1–4]) and spatially-time (in depth and propagation time, [14, 15]) distributions of the acoustic intensity at long and super long distances in the ocean waveguide.

One group of works [1, 2, 11, and 12] states that, for forming the weakly diverging bundles, it is sufficient that smooth extremes should exist in the dependence of the cycle length of the geometrical rays on their departure angles at the source. In the second group of works [7, 13], the necessary and sufficient conditions for forming such bundles are formulated as the existence of extremes in the dependence of the cycle length of the geometrical rays on the ray parameter that is inversely proportional to the phase velocity of sound propagation along the corresponding rays. According to [7, 13], principal contradictions of the two statements exhibit themselves only for the so called caustic bundle that forms around the ray leaving the source in the horizontal direction. Therefore, it seems to be highly important to find the most justified and correct formulation of the conditions for forming the weakly diverging ray bundles [1, 2, 7, and 13].

In addition, an evident question arises about the possibility for the weakly diverging bundles to form if there are no extremes in the dependence of the cycle length of the geometrical rays on the ray parameter, for instance. The fact that the fist derivative of the corresponding function is equal to zero at certain value of its argument can mean the existence of an inflexion point rather than an extreme of the function. Therefore, it is highly important to reveal whether the weakly diverging bundles can form when an inflexion point exists in the dependence of the length of the ray cycle on the ray parameter.

In considering the aforementioned problems, just as in [7], the main attention is paid to obtaining explicit relations that allow one to formulate sufficiently general conclusions.

PARAMETER OF DIFFRACTION DIVERGENCE OF MULTI-MODE BUNDLES IN THE WKB APPROXIMATION

Let us show that the physically correct formulation of the conditions for forming the weakly diverging bundles in vertically stratified oceanic waveguides should involve derivatives of the cycle length of the Brilloin wave (rays) just with respect to the ray parameter. To do so, let us consider propagation of multimode acoustic bundles with sufficiently narrow spatial spectra A_l of the ray-forming modes with numbers $l = [1, L]$. In oceanic waveguides, such bundles can be excited by both point sources [8, 9] and those distributed in depth [16, 17]. In the first case, a multi-bundle structure of the sound field forms [8, 9], while a single bundle can be selectively excited [16, 17] in the second case.

Thus, let us suppose that the spectrum of the modes that corresponds to the bundle with certain dependence of the excitation factors A_l on the mode number l is concentrated in the vicinity of the mode with number $l = l_0$ and has the width $\Delta l = L_0$:

$$\begin{cases} \left|\left|\dfrac{A_l}{A_{l_0}}\right|\right|_{l = l_0 \pm L_0/2} \ll 1, \\ L_0/L \ll 1, \quad l_0/L \ll 1. \end{cases} \quad (1)$$

Here, $L_0 \gg 1$ is the number of efficiently excited modes and $L \gg 1$ is the total number of modes in the waveguide.

Let us show that the diffraction divergence of such a bundle is determined by different-order derivatives of the cycle length of the Brilloin waves with respect to the ray parameter at $l = l_0$, those derivatives characterizing the aberration effects of the corresponding orders. To do so, let us use the results of works [18, 19], which allow one to come to a natural statement that the diffraction divergence of the corresponding mode group (1) depends on a certain dimensionless quantity ζ. In waveguides that are uniform along the propagation path, the latter quantity should be proportional to the horizontal distance r passed by the bundle and to the number L_0 of efficiently excited modes. In addition, the value of ζ should be inversely proportional to the minimal value $R_{\min}$ of the spatial period of reforming the interference structure R_g (diffraction focusing) of the sound field [18]:

$$R_{\min} = \min\{R_g(l, l'; n, n')\}, \quad (2)$$

$$R_g(l, l'; n, n') = \frac{R_{l,l'}R_{n,n'}}{R_{n,n'} - R_{l,l'}}, \quad (3)$$

$$R_{l,l'} = \frac{2\pi}{k_l - k_{l'}} = \frac{2\pi}{k_0(\beta_l - \beta_{l'})}. \quad (4)$$

Here, $R_{l,l'}$ is the spatial period of interference for modes with numbers l and l', k_l is the mode's wave numbers, $\beta_l = k_l/k_0$ is the ray parameter of the corresponding Brillouin waves, $k_0 = \omega/c_0$, ω is the cyclic frequency of sound waves, and $c_0 = \min\{c(z)\} = c(z_0)$ is the minimal value of the sound speed that corresponds to the position $z = z_0$ of the channel axis.

The aforementioned features of the dimensionless variable ζ lead to the following expressions:

$$\zeta = r\mathrm{P}_d, \quad (5)$$

$$\mathrm{P}_d = L_0/|R_{\min}|. \quad (6)$$

According to Eq. (5), the larger the value of P_d (6), which is actually the diffraction parameter of the bundle, the higher the rate of its diffraction divergence.

The large-scale interference structure of the sound field that is most stable at long distances is of the most interest. Such a structure is formed by modes with adjacent ordinal numbers [18]. Therefore, we specify $l' = l + 1$ and $n' = n + 1$ in analyzing Eqs. (2)–(4). If, in addition, one uses condition (1), which expresses the narrowness of the spatial spectrum of the excited modes, it is sufficient to specify $l = l_0 - L_0/2$ and $n = l_0 + L_0/2$ in Eq. (2) for $R_{\min}$ to correctly estimate quantity R_g (3). Then, in view of the adopted assumptions, one can perform the transformation that is analogous to that used in [18], namely, the expansion of all quantities appearing in Eq. (3) for R_g with respect to $(l - l_0)/L \ll 1$ up to the third order of smallness to obtain the following expression for P_d (6):

$$\mathrm{P}_d \approx \left|\mathrm{P}_d^{(2)} - \frac{1}{2}\mathrm{P}_d^{(3)} + \frac{1}{6}\mathrm{P}_d^{(4)}\right|, \quad (7)$$

$$\mathrm{P}_d^{(k)} = L_0^2/R_d^{(k)}. \quad (8)$$

In Eq. (8), the spatial periods $R_d^{(k)}$ of diffraction focusing the sound field, which characterize the scale of aberration effects of the corresponding orders ($k = 2$, 3, and 4), are given by the following expressions [18]:

$$R_d^{(2)} = \frac{k_0}{2\pi}\left\{D_l^3/\frac{dD_l}{d\beta_l}\right\}_{l = l_0}, \quad (9)$$

$$R_d^{(3)} = \left(\frac{k_0}{2\pi}\right)^2\left\{D_l^4/\left[\frac{d^2D_l}{d\beta_l^2} - \frac{3}{D_l}\left(\frac{dD_l}{d\beta_l}\right)^2\right]\right\}\Bigg|_{l = l_0}, \quad (10)$$

$$R_d^{(4)} = \left(\frac{k_0}{2\pi}\right)^3\left\{D_l^5/\left[\frac{d^3D_l}{d\beta_l^3} - \frac{10}{D_l}\left(\frac{d^2D_l}{d\beta_l^2}\right)\left(\frac{dD_l}{d\beta_l}\right) + \frac{15}{D_l^2}\left(\frac{dD_l}{d\beta_l}\right)^3\right]\right\}\Bigg|_{l = l_0}. \quad (11)$$

Here,

$$D_l = 2\beta_l\int_{z_B}^{z_H} dz/\gamma_l(z) \quad (12)$$

is the cycle lengths of the Brilloin waves that reach the minimal $z_B(\beta_l)$ and maximal $z_H(\beta_l)$ depths in the water layer, respectively,

$$\gamma_l(z) = \sqrt{n^2(z) - \beta_l^2},$$

and $n(z) = c_0/c(z)$ is the refraction index of the sound waves.

In describing ordinary multi-mode bundles, the reference Brillouin rays for which $\beta_l = \beta_{l_0}$ obey the condition

$$\left(\frac{dD_l}{d\beta_l}\right)\Bigg|_{\beta_l = \beta_{l_0}} \neq 0, \quad (13)$$

and we can restrict ourselves by the only first summand $P_d \approx |P_d^{(2)}|$ in Eq. (7) corresponding to the aberration effects of the second order in those bundles.

To describe propagation of the first-type weakly diverging bundles with the Brilloin rays obeying the condition

$$\left(\frac{dD_l}{d\beta_l}\right)\bigg|_{\beta_l = \beta_{l_0} = \beta_c} = 0, \quad \left(\frac{d^2D_l}{d\beta_l^2}\right)\bigg|_{\beta_l = \beta_{l_0} = \beta_c} \neq 0, \qquad (14)$$

of an extreme existing in the dependence $D_l(\beta_l)$ at $\beta_l = \beta_{l_0} = \beta_c$, it is sufficient to account for the second summand $P_d \approx \frac{1}{2}|P_d^{(3)}|$ in Eq. (7), that summand characterizing the aberration effects of the third order. However, if the following conditions are satisfied for the reference Bruilloin rays:

$$\left(\frac{dD_l}{d\beta_l}\right)\bigg|_{\beta_l = \beta_{l_0} = \beta_c} = \left(\frac{d^2D_l}{d\beta_l^2}\right)\bigg|_{\beta_l = \beta_{l_0} = \beta_c} = 0, \qquad (15)$$
$$\left(\frac{d^3D_l}{d\beta_l^3}\right)\bigg|_{\beta_l = \beta_{l_0} = \beta_c} \neq 0$$

that correspond to the existence of an inflexion point in the dependence $D_l(\beta_l)$ at $\beta_l = \beta_{l_0} = \beta_c$, Eqs. (9)–(11) lead to a conclusion that the second-type weakly diverging are a generated form in oceanic waveguides. Such bundles are mainly influenced by the aberration effects of the fourth order, $P_d \approx \frac{1}{6}|P_d^{(4)}|$.

Thus, in vertically stratified oceanic waveguides, weakly diverging multi-mode bundles can form when not only sufficiently smooth extremes (14) [7, 13] but also inflexion points (15) exist in the dependence of the cycle length of the Brillion waves (12) on the ray parameter that is determined by the dispersion equation:

$$k_0 \int_{z_B}^{z_H} \gamma_l(z)dz = \pi(l - \nu). \qquad (16)$$

Such a parameter is obtained with the use of the WKB approximation [20, 21]. Parameter ν in Eq. (16) takes values of 1/2 or 1/4 depending on whether depths z_B and z_H correspond to the turning horizon for the Brilloin waves or perfectly reflecting interfaces of the media [20, 21].

It is worth mentioning that the approximate expressions for $P_d \approx |P_d^{(2)}|$, Eqs. (7) and (8), which follow from Eq. (6) for the ordinary bundle, differs from the exact expression obtained in [17] for P_d in the case of Gaussian bundle. However, such a difference consists in nothing but a constant factor of $\pi/4$.

WEAKLY DIVERGING BUNDLES OF RAYS IN GEOMETRICAL ACOUSTICS

Now, let us show that, for weakly diverging bundles of geometrical rays to form in vertically stratified oceanic waveguides, it is necessary and sufficient to satisfy the same conditions as Eqs. (14) and (15) in which, however, D_l and β_l should be substituted by cycle lengths D and ray parameter β corresponding to the rays of geometrical acoustics.

To do so, let us use the approach proposed in [22] for obtaining the approximation of geometrical acoustics for the sound field in waveguides. Then, by using the WKB approximation for the mode representation of the sound field and replacing the summation over l by integration over β_l with the stationary-phase method [7, 22], one can obtain the well known expressions [20, 21] for the intensity J_β at the geometrical ray and the number Δl_β of the modes efficiently forming the ray:

$$J_\beta = (p_0R_0)^2\left\{\frac{\beta/\gamma(z_s, \beta)\gamma(z, \beta)}{r_\beta|\partial r_\beta/\partial\beta|}\right\}, \qquad (17)$$

$$\Delta l_\beta = D\sqrt{\frac{k_0}{2\pi}\Big/\left|\frac{\partial r_\beta}{\partial \beta}\right|} = \sqrt{\frac{|R_d^{(2)}|}{D}\left|\frac{dD/d\beta}{\partial r_\beta/\partial\beta}\right|}. \qquad (18)$$

Here,

$$r_\beta = mD(\beta) + \bar{\mu}D(z_s, \beta) + \mu D(z, \beta) \qquad (19)$$

is the horizontal distance passed by the geometrical ray, $m = 0, 1, \ldots$ is the integer number of its cycles, z_s is the depth of the point source, $\beta = \cos\chi_0$, and $\chi_0 = \chi(z_0)$ is the grazing angle of the ray at the channel axis.

Other values appearing in the equations are as follows:

$$D(\beta) = 2\beta\int_{z_B}^{z_H} dz/\gamma(z, \beta), \quad D(z, \beta) = \beta\int_{z_B}^{z} dz/\gamma(z, \beta), \qquad (20)$$
$$R_d^{(2)} = \frac{k_0}{2\pi}D^3/(dD/d\beta), \quad \gamma(z, \beta) = \sqrt{n^2(z) - \beta^2},$$

$$\bar{\mu} = \begin{cases} -1, & \chi_s < 0, \\ +1, & \chi_s > 0, \end{cases} \quad \mu = \begin{cases} +1, & \chi < 0, \\ -1, & \chi > 0, \end{cases} \qquad (21)$$

with p_0 being the amplitude of the pressure generated by the point source in a uniform medium at distance R_0. In Eq. (21), $\chi_s = \chi(z_s)$ is the grazing angle of the ray at the source ($\chi_s < 0$ and $\chi_s > 0$ correspond to downward and upward directions, respectively) and $\chi = \chi(z)$

is the grazing angle of the ray at the depth of reception z, so as

$$\sin\chi_s = \bar{\mu}\gamma(z_s, \beta)/n(z_s), \quad \sin\chi = -\mu\gamma(z, \beta)/n(z).$$

According to Eq. (17) (see also [20, 21]), the extremely maximal values of the acoustic intensity at the rays ($1/J_\beta = 0$) are reached at the points of touching the caustics, z_c and r_c. In r_β of Eq. (19), such points correspond to certain values of the ray parameter $\beta = \beta_{ca}(z_s, z_{ca})$, which are the solutions of the equation

$$(\partial r_\beta/\partial\beta)|_{\beta = \beta_{ca}} = 0. \tag{22}$$

Equation (18) shows that, along the ray trajectory, the sound field at the ray is formed by the extremely maximal number of constructively interfering modes just at the points of touching the caustics ($1/\Delta l_\beta = 0$).

Here, the matter of our interest is a rather evident condition of forming the caustics in vertically stratified oceanic waveguides. Such a condition consists in the existence of a sufficiently smooth extreme in the dependence $D(\beta)$ at a certain value $\beta = \beta_c$:

$$\left(\frac{dD}{d\beta}\right)\Bigg|_{\beta = \beta_c} = 0, \quad \left(\frac{d^2D}{d\beta^2}\right)\Bigg|_{\beta = \beta_c} \neq 0. \tag{23}$$

Thus, in view of Eqs. (19) and (23) for the rays with $\bar{\mu} = -\mu$, Eq. (22) can be used to find the solution $\beta_{ca} = \beta_c$, to which certain coordinates of the points correspond,

$$z_c = z_s, \quad r_c = mD(\beta_c), \tag{24}$$

at the caustic lines on the $z - r$ plane. However, if rays of an arbitrary type are considered, that is, when $\bar{\mu} = -\mu$, $z_s \neq z$, $\bar{\mu} = \mu$, $z_s \neq z$, and $z_s = z$, one can obtain only approximate solutions of Eq. (22) in their general forms. If one uses the expansion of the derivative $\partial r_\beta/\partial\beta$ into series with respect to $\Delta\beta = \beta_{ca} - \beta_c$ to the second order of smallness in the vicinity of $\beta = \beta_c$, Eq. (22) yields a quadratic equation for $\Delta\beta$ whose solution has the following form:

$$\Delta\beta_1^{(\pm)} = -\frac{a_2}{2a_3}\left(1 \pm \sqrt{1 - \frac{4a_1a_3}{a_2^2}}\right), \tag{25}$$

where

$$a_1 = \left(\frac{\partial r_\beta}{\partial\beta}\right)\Bigg|_{\beta = \beta_c}, \quad a_2 = \left(\frac{\partial^2 r_\beta}{\partial\beta^2}\right)\Bigg|_{\beta = \beta_c},$$
$$a_3 = \frac{1}{2}\left(\frac{\partial^3 r_\beta}{\partial\beta^3}\right)\Bigg|_{\beta = \beta_c}. \tag{26}$$

Naturally, solutions of Eq. (25), first, should be real,

$$4\frac{a_1a_3}{a_2^2} < 1, \tag{27}$$

and, second, they must belong to the range $\Delta\beta_c$ that is characteristic for the weakly diverging bundle (see below):

$$\left|\Delta\beta_1^{(\pm)}\right| \le \Delta\beta_c. \tag{28}$$

Note that quantity $\Delta\beta_1^{(-)}$ is one that can preferably obey condition (28). The minimal value of that quantity

$$\Delta\beta_1^{(-)} \approx -a_1/a_2$$

will be reached when

$$4|a_1a_3|/a_2^2 \ll 1.$$

It is quite natural that, for the rays forming the weakly diverging bundles, quantities J_β (17) and Δl_β (18) should take the maximally allowable values along the trajectories of the rays on the z–r plane. Therefore, the necessary and sufficient conditions for bundles of weakly diverging rays to be formed in oceanic waveguides are also for Eqs. (23), which characterize the existence of smooth extremes in the dependence $D(\beta)$ (20).

The aforementioned features of dependences $J_\beta(z, r)$ (17) and $\Delta l_\beta(z, r)$ (18) lead to a conclusion that the weakly diverging bundles do not predominate in their intensity in comparison with the total sound field in the waveguides because the spatial structures corresponding to the caustic lines [13] have the highest intensity. Nevertheless, the rays forming such bundles also participate in forming certain parts of the caustic lines that become closer to each other as the number m of ray cycles increases. As a result, according to Eqs. (19) and (25) and the data of numerically modeling [13], specific caustic bundles form, which predominate in their intensity and correspond to weakly diverging bundles. The characteristic width $\Delta\beta_c$ of the range within which the ray parameter varies,

$$\beta_c - \frac{\Delta\beta_c}{2} \le \beta \le \beta_c + \frac{\Delta\beta_c}{2},$$

for the rays forming the parts of caustic lines corresponding to the caustic bundle is determined by the following expression:

$$\Delta\beta_c = |\beta_{ca}(z_{Bc}) - \beta_{ca}(z_{Hc})|, \tag{29}$$

where

$$z_{Bc} = z_B(\beta_c), \quad z_{Hc} = z_H(\beta_c).$$

Here, the range

$$\beta_c \le \beta \le \beta_c + \Delta\beta_c/2$$

corresponds to the upward parts and the range

$$\beta_c - \frac{\Delta\beta_c}{2} \le \beta \le \beta_c$$

corresponds to the downward parts of the caustics.

With the use of the approach similar to that of [22], we find the following expression for correctly determining the sound field in the vicinity of caustics, namely, at the rays that touched or did not touch caustics, and the number of modes efficiently forming such rays:

$$J_\beta(\beta) = \left\{ \beta/\gamma(z_S, \beta)\gamma(z, \beta) r_\beta \sqrt{\frac{|r_\beta - r_c|}{2}} \left| \frac{\partial^2 r_\beta}{\partial \beta^2} \right| \right\} \Bigg|_{\beta \to \beta_{ca}}, \quad (30)$$

$$\Delta l_\beta(\beta) = \left\{ 3\left(\frac{k_0}{2\pi}\right)^2 \frac{D^3(\beta)}{|\partial^2 r_\beta/\partial\beta^2|} \right\}^{1/3} \Bigg|_{\beta \to \beta_{ca}}, \quad (31)$$

where

$$r_c = r_\beta(z_c, \beta_{ca}),$$

$$\frac{1}{2}\left\{\frac{3}{k_0^2}\left|\frac{\partial^2 r_\beta}{\partial\beta^2}\right|\right\}^{1/3}\Bigg|_{\beta = \beta_{ca}} \ll |r_\beta - r_c| \ll \left\{\left|\frac{\partial^2 r_\beta}{\partial\beta^2}\right|^3 \Big/ \left(\frac{\partial^3 r_\beta}{\partial\beta^3}\right)^2\right\}\Bigg|_{\beta = \beta_{ca}}.$$

Equations (30) and (31) show that, in the vicinity of caustic lines on the $z - r$ plane (22), the extremely maximal intensity of the sound field ($1/J_\beta(\beta_{ca}) = 0$) at the rays and the number of modes efficiently forming those rays ($1/\Delta l_\beta(\beta_{ca}) = 0$) will be reached if the equality that is complementary to Eq. (22) is met:

$$\left(\frac{\partial^2 r_\beta}{\partial \beta^2}\right)\Bigg|_{\beta = \beta_{ca}} = 0. \quad (32)$$

It is quite evident that the conditions similar to those of Eq. (15) are necessary and sufficient for the equalities (22) and (32) that determine forming weakly diverging bundles of the second type to be commonly valid:

$$\left(\frac{dD}{d\beta}\right)\Bigg|_{\beta = \beta_c} = \left(\frac{d^2 D}{d\beta^2}\right)\Bigg|_{\beta = \beta_c} = 0, \quad (33)$$
$$\left(\frac{d^3 D}{d\beta^3}\right)\Bigg|_{\beta = \beta_c} \ne 0.$$

Those expressions correspond to the existence of an inflexion point in the dependence $D(\beta)$ (20) at $\beta = \beta_c$.

For a certain type of the rays with $\bar{\mu} = -\mu$ at $z_s = z$, the value $\beta = \beta_c$ is the exact solution $\beta_{ca}(z_S) = \beta_c$ of Eq. (32). Therefore, the coordinates of the points at which such rays touch the caustics can be found from Eq. (24). However, only approximate solutions of Eq. (32) can be found in their general form if rays of arbitrary types are considered, that is, if $\bar{\mu} = -\mu$, $z_S \ne z$ or $\bar{\mu} = \mu$, $z_S \ne z$, and $z_S = z$. Thus, the expansion of the derivative $\partial^2 r_\beta/\partial\beta^2$ in series with respect to $\Delta\beta = \beta_{ca} - \beta_c$ in the vicinity of $\beta = \beta_c$ to the second order of smallness with the use of Eq. (32) yields a quadratic equation for $\Delta\beta$ whose solution has the form

$$\Delta\beta_2^{(\pm)} = -\frac{b_2}{2b_3}\left(1 \pm \sqrt{1 - \frac{4b_1b_3}{b_2^2}}\right), \quad (34)$$

where

$$b_1 = a_2, \quad b_2 = 2a_3, \quad b_3 = \frac{1}{2}\left(\frac{\partial^4 r_\beta}{\partial\beta^4}\right)\Bigg|_{\beta = \beta_c}. \quad (35)$$

The solutions $\Delta\beta_2^{(\pm)}$ (34) should also satisfy the conditions that are analogous to those of Eqs. (27) and (28):

$$4b_1b_3/b_2^2 \le 1, \quad (36)$$
$$|\Delta\beta_2^{(\pm)}| \le \Delta\beta_c.$$

According to Eqs. (33)–(35), condition (36) can be preferably met with quantity $\Delta\beta_2^{(-)}$ whose minimal values

$$\Delta\beta_2^{(-)} \approx -b_1/b_2$$

will be reached when the following condition is satisfied:

$$4|b_1b_3|/b_2^2 \ll 1.$$

It is worth mentioning that the rays of such a weakly diverging bundle (33) form certain parts of the caustic lines, namely, the cuspidal points of the caustics (see [20]), whose coordinates z_c and r_c satisfy the equality that follows from Eqs. (22) and (32):

$$\Delta\beta_2^{(-)} = \Delta\beta_1^{(+)}. \quad (37)$$

The use of $\Delta\beta_1^{(+)}$ in Eq. (37) is caused by the fact that, in this case, this value is the only one that can preferably satisfy condition (28).

FORMING THE CAUSTIC RAY BUNDLES

As we have mentioned earlier, a specific caustic bundle forms [13] if conditions (28) or (36), (37) are met at the corresponding parts of the caustic lines that are envelopes of the set of rays forming the weakly diverging bundle, those rays becoming closer to each other when the horizontal distance increases.

However, in stratified oceanic waveguides, caustics can exist with no extremes (23) or inflexion points (33) in the dependence $D(\beta)$ (20). In this case, the presence of caustics is known [20, 21] to be caused by the existence of lower, $\gamma(z_H, \beta) = 0$, upper, $\gamma(z_B, \beta) = 0$, or both lower and upper, $\gamma(z_H, \beta) = \gamma(z_B, \beta) = 0$, horizons of turning the rays. Therefore, it is strongly important to reveal the conditions whose satisfaction determines forming the caustic bundle, which is similar to that formed by rays of a weakly diverging bundle.

To solve the aforementioned problem, let us analyze the behavior of quantities J_β (17) and Δl_β (18) for the rays with ray parameters that are close to the value $\beta = \beta_S = n(z_S)$, which is characteristic for the ray horizontally leaving the source. In turning points of rays, $n(z) = \beta_H = \beta$, the following condition is satisfied:

$$0 < \left\{\gamma(z, \beta)\left|\frac{\partial r_\beta}{\partial \beta}\right|\right\}\Bigg|_{\beta = \beta_H} < \infty. \quad (38)$$

The coincidence of the depth of sound transmission with that of reception, $z = z_S$, and with the turning horizon for the ray horizontally leaving the source, $\beta_H = \beta_S$, is determined by the following equality:

$$\gamma(z, \beta_H) = \gamma(z_S, \beta_H) = 0.$$

If the latter equality is valid, the intensity of the sound field will reach its extremely maximal value at some points of the waveguide ($1/J_\beta = 0$). In this case, the number of modes efficiently forming the ray with $\beta = \beta_S$ is limited by a finite value: $0 < \Delta l_\beta < \infty$.

Thus, the ray bundle that is not compulsorily a weakly diverging one,

$$\left(\frac{dD}{d\beta}\right)\Bigg|_{\beta = \beta_S} \neq 0,$$

can form the caustic bundle around the ray horizontally leaving the source. However, in the case at hand, one can hardly obtain the approximate solutions that are analogous to Eqs. (25) or (34), in which β_c is substituted by β_S. Such a situation is caused by the fact that one cannot correctly expand the derivative appearing at the right-hand side of the equation of caustics (22) into series in the vicinity of values $\beta = \beta_S$ or $\beta = \beta_H$ because, according to Eqs. (19) and (20), those derivatives become infinite. The latter statement whose consequence is relation (38) can be explicitly verified by using the simplest model of the sound channel with a linear dependence of $n^2(z)$ [20].

According to the aforementioned features, revealing the possibility for such a caustic bundle to exist and determining the values of the ray parameter β_{ca} corresponding to certain parts of the caustic lines forming the bundle requires Eq. (22) to be solved each time when particular dependences $n^2(z)$ are specified. Nevertheless, the study [7] where different-type dependences $n^2(z)$ were used allowed one to formulate rather general conditions

$$\left(\frac{dD}{d\beta}\right)\Bigg|_{\beta = \beta_S} < 0, \quad \left(\left|\frac{dD}{d\beta}\right|\right)\Bigg|_{\beta = \beta_S} < \infty, \quad (39)$$

whose satisfaction is necessary for the caustic bundle around the ray horizontally leaving the source to form in a vertically stratified waveguide. The Snell law [21] can be used to obtain the differential relation

$$\frac{dD}{d\beta} = -\frac{dD}{d\chi_S}/n(z_S)\sin(\chi_S). \quad (40)$$

The latter relation leads to a conclusion that, for forming the caustic bundle determined by conditions (39), the existence of the dependence $D(\chi_S)$ that does not increase at $\chi_S \longrightarrow 0$ is necessary and sufficient.

It is important to mention that, according to Eq. (39), the following equality is always valid for the waveguides with finite values of the derivative $(dD/d\beta)|_{\beta = \beta_S}$:

$$\left(\frac{dD}{d\chi_S}\right)\Bigg|_{\chi_S = 0} = 0. \quad (41)$$

Such equality is equivalent to the condition of existing weakly diverging ray bundles, which follows from Eqs. (23) and (40), as it was formulated in [1]:

$$(dD/d\chi_S)|_{\chi_S = \chi_c} = 0, \quad \chi_c = \arccos(\beta_c/n(z_s)) \neq 0.$$

CAUSTIC AND WEAKLY DIVERGING RAY BUNDLES IN THE CANONIC UNDERWATER SOUND CHANNEL

The above conclusions on forming the weakly diverging and caustic bundles allow one to correctly interpret the data obtained in [12]. To that end, in analogy with [12], let us consider an oceanic waveguide with the canonic underwater sound channel [23]:

$$\begin{aligned} c(z) &= c_0[1 + \varepsilon(e^{\eta} - \eta - 1)], \quad 0 \le z \le H, \\ \eta &= 2(z_0 - z)/B, \quad z_0 = B = 1.3\ \text{km}, \\ c_0 &= 1.492\ \text{km/s}, \quad \varepsilon = 1.14 \times 10^{-2}B/2. \end{aligned} \quad (42)$$

In spite of the fact that such a channel does not satisfy the condition (41) of Brekhovskikh, the data of numerical calculations [12] of the ray trajectories show that a weakly diverging ray bundle forms around the ray horizontally leaving the source.

However, one can easily find that, first, condition (41) of Brekhovskikh and, second, condition (39) of forming the caustic bundle around the ray horizontally leaving the source are both met in channel (42).

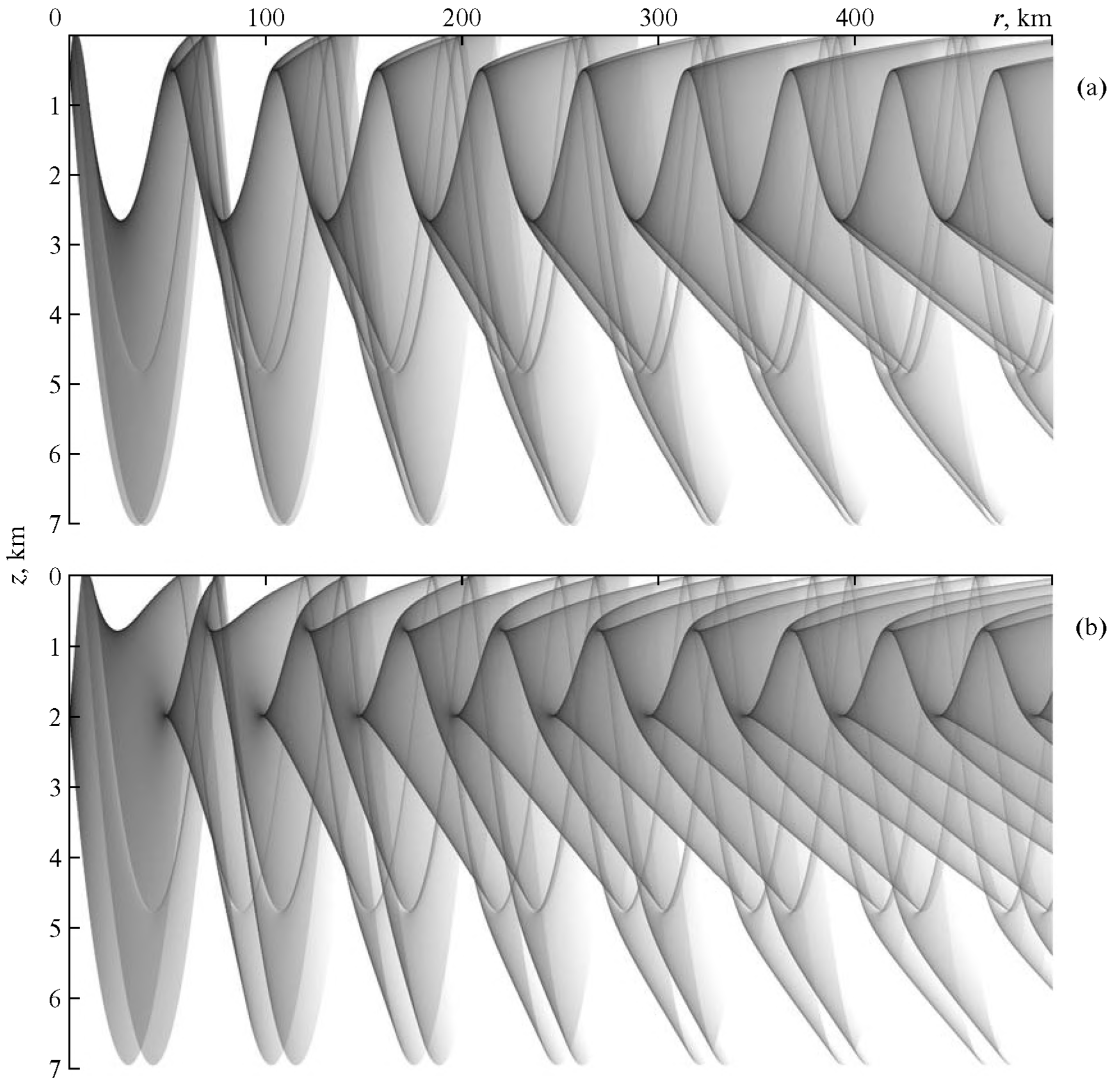

Fig. 1. Spatial distributions of intensity $J_0(z, r)$ as functions of depth z and horizontal distance r for the sound field produced by refracted rays and those interacting with the pressure-release surface. The distributions are normalized to the cylindrical spread of the wave front. The shown dependences are calculated with averaging over $\Delta z = 6.5$ m and $\Delta r = 200$ m at $z_S =$ (a) 0.5 and (b) 2 km with the use of the geometrical approximation [24, 25] for the canonic underwater sound channel (42).

Naturally, let us restrict ourselves by considering the behavior of the dependence $D(\beta)$ (20) within the range of the ray parameter where the following relation is valid:

$$\frac{1}{\varepsilon}\left(\frac{1}{\beta} - 1\right) \ll 1. \tag{43}$$

For such a channel, Eq. (20) can be used to obtain an approximate expression [21] for the cycle length of refracted rays to the second order of smallness with respect to the corresponding parameter (42):

$$D(\beta) \approx D_0\left\{1 + \frac{1}{12\varepsilon^2}\left(\frac{1}{\beta} - 1\right)^2\right\}, \qquad D_0 = \pi B/\sqrt{\varepsilon}. \tag{44}$$

Then, with the use of Eq. (44), the required derivatives can be found:

$$\frac{dD}{d\beta} \approx -\frac{D_0}{6\varepsilon^2\beta^2}\left(\frac{1}{\beta} - 1\right), \quad \frac{d^2D}{d\beta^2} \approx \frac{D_0}{6\varepsilon^2\beta^3}\left(\frac{3}{\beta} - 2\right). \tag{45}$$

The latter expressions show that condition (39) and, hence, condition (41) of Brekhovskikh is valid.

In addition, Eqs. (23) and (45) show that, in the canonic sound channel (42), the weakly diverging bundle of the refracted rays forms only if the source is at the channel axis, $z_S = z_0$. Such a bundle forms around the ray horizontally leaving the source, which is the caustic ray. With other positions of the source, $z_S \neq z_0$, as considered in [12], the caustic bundle produced by certain parts of caustic lines forms around the horizontal ray. That fact is illustrated by the Fig. 1, in which the spatial distribution for intensity $J_0(r, z) = rJ(r, z)$ of the sound field, is presented in the form of density of grey (with a dynamic range of 30 dB), the distribution being normalized to the cylindrical spread of the wave front. The presented data are obtained by numerical calculations with the use of the modified approximation [24, 25] of geometrical acoustics for two substantially different depths of the source, $z_S = 2$ km (as in [12]) and $z_S = 0.5$ km.

According to the figure, the dependences $J_0(r, z)$ decisively and unambiguously prove that only the caustic bundle forms around the ray horizontally leaving the source at $z_S \neq z_0$, in contrast to the dependences obtained in [12]. That ray consists from certain parts of the caustic lines that become closer to each other as distance r increases.

Naturally, if one accounts for the contribution of rays interacting with the pressure-release surface into $J_0(r, z)$, a weakly diverging bundle also forms in such a waveguide because of the existence of a sufficiently smooth minimum in the dependence $D(\beta)$ [8] within the range $n(H) < \beta < n(0)$. The rays of that bundle produce certain parts of the caustic lines that form the corresponding caustic bundle (see the Fig. 1).

CONCLUSIONS

To conclude with, let us consider the main results of this work and their consequences.

For the bundles with a sufficiently narrow spectrum of the modes forming them, a concept of diffraction parameter (6) is introduced. That parameter is proportional to the width of the spectrum and inversely proportional to the minimal spatial period of the diffraction focusing of the sound field in the waveguide. The analysis of the diffraction parameter allows one to determine that, for weakly diverging acoustic bundles to form in vertically stratified oceanic waveguides, it is necessary and sufficient that either smooth extremes (14) or inflexion points (15) should exist in the cycle length of the Brillouin waves as a function of their ray parameter that is inversely proportional to the phase velocity of the corresponding modes. In a rather rare latter case, the first and second derivatives of the aforementioned dependence should simultaneously go to zero.

By using the approximation of geometrical acoustics, it is shown that, for the formation of weakly diverging acoustic bundles in vertically stratified oceanic waveguides, the satisfaction of conditions (22) and (33) that are analogous to those of Eqs. (14) and (15) in the dependence of the length of ray cycles on the ray parameter is necessary and sufficient. Similarly, it is proven that the rays of the weakly diverging bundle form certain parts of caustic lines producing the so called caustic bundle that predominates in its intensity. It is also determined that, in waveguides of certain types (39) without the weakly diverging bundles, the caustic bundles form around the rays that leave the source in the horizontal direction.

The aforementioned results allow one to come to a conclusion that the statements made in [26] are incorrect, those statements consisting in that the mode approach lead to a more strict necessary condition than Eqs. (14) and (22) for the weakly diverging bundle to form, namely, the condition of the linearity of the horizontal wave number as a function of the mode's ordinal number. To prove that incorrectness, let us expand the ray parameter of the Brillouin waves into series in the vicinity of the value $l = l_0$ that corresponds to the maximum in the spectrum of the excited modes (1):

$$\beta_l \approx \sum_{n=0}^{N} \left(\frac{d^n \beta_l}{dl^n}\right)_{l=l_0} \frac{(l-l_0)^n}{n!}. \tag{46}$$

Here, N is the number of summands that should be accounted for to correctly describe propagation of the bundle. Then, the differential relation

$$\frac{d\beta_l}{dl} = -\frac{2\pi}{k_0 D_l}$$

that follows from dispersion equation (16) can be used to bring Eq. (46) to a more useful form:

$$\beta_l \approx \beta_{l_0} - \frac{2\pi}{k_0} \sum_{n=1}^{N} \frac{(l-l_0)^n}{n!\, R_d^{(n)}}, \tag{47}$$

where

$$R_d^{(n)} = \left\{\left(-\frac{2\pi}{k_0}\right)^{n-1} \left(\frac{1}{D_l}\frac{d}{d\beta_l}\right)^{(n-1)} \left(\frac{1}{D_l}\right)\right\}\Bigg|_{l=l_0}^{-1}$$

are the characteristic scales of aberration effects of the corresponding orders, $n \geq 2$, see Eqs. (9)–(11).

If conditions (14) and (15) of forming the weakly diverging acoustic rays are met and the aberration effects of orders $N = 2$ and $N = 3$ are accounted for, Eq. (47) can be used with the value $l_0 = l_c(\beta_c)$ corresponding to quantity β_c to find the approximate linear dependence

$$\beta_l \approx \beta_{lc} - \frac{2\pi}{k_0 D_{lc}}(l - l_c)$$

that characterizes the statement made in [26].

It is also worth mentioning that the WKB approximation for the mode representation of the sound field in the waveguide nevertheless leads to the additional condition formulated in [8] for the caustic bundle produced by the Brillouin waves of the weakly diverging bundle to pronouncedly exhibit itself in the spatial distribution of the sound field. In fact, such a condition consists in that the point source of the tonal signal should excite a sufficiently large number of mode bundles with different values of l_0 within the range $\Delta\beta_c$ (29) of the ray parameter of the Brilloin waves, those values of l_0 being characteristic for the caustic bundle. A similar additional condition is formulated in [8] for the caustic bundle formed around the ray horizontally leaving the source to noticeably exhibit itself.

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