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Propagation of Elastic Waves in Layered Composites with Microdefect Concentration Zones and Their Simulation with Spring Boundary Conditions

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Abstract—The possibility is studied of applying spring boundary conditions to describe propagation of elastic waves in layered composites with nonperfect contact of components or in the presence of groups of microdefects at the interface. Stiffnesses in spring boundary conditions are determined by crack density, the average size of microdefects, and the elastic properties of the materials surrounding them. In deriving the values of the effective stiffness parameters, the Baik–Thompson and Boström–Wickham approaches are applied, as well as the integral approach. The components of the stiffness matrices are derived from consideration of an incident, at a random angle to the interface, plane wave in the antiplane case, and at a normal angle in the plane case. The efficiency of this model and the possibility of using its results in the three-dimensional case are discussed.

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1. INTRODUCTION

A precursor to destruction of a sample of a composite or homogeneous material is usually the formation of microcracks; their subsequent growth into macrocracks makes further operation impossible or dangerous. The occurrence of cracks probably both at the manufacturing stage and during operation are a consequence of material wear close to critical loads, etc. The occurrence of new composite materials only increases the urgency in detecting internal inhomogeneities and restoring their parameters [1, 2], in particular, detecting internal and surface defects by ultrasonic nondestructive control [3, 4], which requires the presence of effective mathematical models describing the diffraction of elastic waves on defects [1]. Of particular interest are delamination or zones of discontinuity and their oscillations under dynamic looding; these are considered in the present article.

The mathematical model of a crack with stress-free faces is frequently used. However, in a number of cases, with respect to a zone of nonperfect contact or a concentration of microdefects, i.e., regions with alternating zones of continuity and discontinuity in displacements, this is insufficient. In practice, oscillations of bodies with cracks, which can be curvilinear or branched, occur quite often with acting of crack faces [5]. In other words, a complete description of a real crack is an extremely difficult mathematical problem, since it requires allowance for the complicated geometry and nonlinear interaction of crack surfaces. Through a zone of nonperfect contact, an energy current of nonzero power takes place, whereas there is no energy flow through crack faces. Therefore, owing to smaller dispersion on delamination, their detection is more complicated than revealing cavities and cracks.

To mathemically describe the dynamic behavior of a crack medium or a zone containing internal defects, it is possible to introduce a distribution of microcracks (cavities) [6–8] or, on the contrary, spots of contact between nonadjoint layers [9]. Other approaches to simulating damaged materials have been developed: replacement of the damaged zone by a thin viscoelastic layer, including combination with the introduction of spring boundary conditions (SBCs) considered in [10, 11]. In any case, to describe the behavior of an elastic medium with delamination, information on the damage (fracturing) in the area is necessary: the number, orientation, sizes of microcracks, etc.

The use of SBCs seems a rather effective tool in simulating damaged interfaces and rough contacting surfaces [11, 12]. First, SBCs are more general boundary conditions than those on a single crack, and they make it possible to describe a wider class of delaminations. Secondly, in simulating delaminations, is often simpler to construct the solution for the SBCs than it is for multiple cracks. Third, in solving problems on identifying defects, SBCs can give knowledge on the sizes of delamination and degrees of damage. To obtain this knowledge, a relation between the constants in SBCs and delamination parameters is necessary, and determination of this relation is the main objective here. Below, SBC constants are derived from consideration of the plane and antiplane problem and their possible further application is discussed in simulating a delaminations interface, including the three-dimensional case.

If in the majority of currently avaiable works, damage to a homogeneous sample has been considered [1, 4, 5, 7–19], in the present work, a generalized theoretical conclusion is considered on the constants in SBCs for the case of a delamination interface at the joint of two different isotropic materials, which was begun in [20]. In deriving the stiffness values for SBCs, an assumption is used on the relative smallness of the characteristic crack size relative to wavelengths of an incident field. Such an assumption can be used for a rather wide range of parameters. Indeed, if the sizes of the cracks forming the delamination are larger than the wavelengths, they can be considered as individual macrocracks because they are potentially dangerous stress concentrators [8].

In this study, first, the SBCs are formulated in a general form, and then the relation of constants with elastic moduli of contacting materials and delamination characteristics is derived. In the antiplane case, a constant is derived from consideration of a plane wave incident at any angle, and it is demonstrated that under assumptions of proportionality of wavelengths and delamination sizes, the stiffness value hardly depends on the angle of incidence. For the plane problem, the normal angle of incidence of P and SV waves is considered in order to avoid considering types of waves that differ from an incident wave. In both cases, the derivation scheme is in many respects similar to the one used in [9]. At the first step, propagation of a plane wave from one half-space to another with a single crack on the interface is considered, and an asymptotics for the jump function of displacement on crack faces is derived. Then the solution for a single crack is generalized to the case of crack distribution by averaging over the ensemble [21]. From the equality of the transmission coefficient for crack distribution and the transmission coefficient for two half-spaces connected by the distributed spring, the dependence of stiffness in SBCs on the damage parameters [9] is derived. In conclusion, the applicability range of these models is analyzed.

2. INTRODUCING SPRING BOUNDARY CONDITIONS

To describe the dynamic behavior of damaged zones, a distributed set of cracks is often used. Here, it is necessary to note a number of works by Achenbach et al. (for example, [13, 14]), where passage of plane waves in space and scattering on different variants of crack distribution has been considered, as well as [8], which includes a review of methods using averaging approaches. In particular, using the example of the reflection coefficient, the results following from approximated theories [15] were compared. As well, a solution has been constructed for the case of identical randomly oriented cracks [16, 17] and cracks whose lengths change according to the normal distribution law [17]. The reflection coefficient in both cases is no more than twice as large as the periodic array. On the whole, the studies indicate a sufficiently small distinction between amplitudes in the far zone with various variants of crack distribution, but with identical damage to the interface. In many cases, this makes it possible to select the simplest variant of defect distribution for each specific problem.

It is quite natural to use such arbitrariness in introducing a simpler, from the mathematical standpoint, spring model, which assumes replacement of the distributed set of the interface cracks by SBCs given on a surface containing or approximating this set. Transition occurs from the assumption of equivalence of amplitudes of the given spring and crack model in a zone far from delamination. In the quasistatic approximation, Baik and Thompson [10] introduced SBCs to describe oscillations of an unlimited damaged surface between two half-spaces. SBCs were also derived by Boström and Wickham [9] from the solution to the problem on oscillations of a partially closed crack. Their efficiency has been demonstrated in a number of works on simulating incomplete contact between surfaces of identical materials (see, e.g., [11, 12, 18]); this was also shown experimentally in [22].

In the general three-dimensional case, SBCs at point $\mathbf{x}$ of surface S with normal $\mathbf{n}$ can be written in the form

$$\boldsymbol{\tau}_{\mathbf{n}}(\mathbf{x}) = \kappa(\mathbf{u}_{\mathbf{n}-}(\mathbf{x}) - \mathbf{u}_{\mathbf{n}+}(\mathbf{x})), \quad \mathbf{x} \in S;$$

κ is a 3×3 stiffness matrix; $\mathbf{n}\pm$ are external and internal normals to the surface at the considered point $\mathbf{x}$; $\boldsymbol{\tau}_{\mathbf{n}}$ are the normal and tangent components of the stress tensor on the area with a normal $\mathbf{n}$ in $\mathbf{x}$; and $\mathbf{u}$ is the displacement vector. In the isotropic case, it is possible to choose a local system of coordinates in $\mathbf{x}$ in such a manner that the three diagonal components of the stiffness matrix remain nonzero:

$$\kappa_{11} = \kappa_x, \quad \kappa_{22} = \kappa_y, \quad \kappa_{33} = \kappa_z.$$

Further expressions are derived from consideration of the case of strip cracks for stiffnesses κ_i via the fracture and elastic properties of materials, i.e., for delaminations strongly extending in one of the directions.

3. SHEARING COMPONENT κ_y (ANTIPLANE CASE)

3.1. Single Interface Crack

We consider first the transmission of a plane elastic SH wave from one half-space to another, with an interface strip-like crack located on their joint. The Cartesian coordinate system is chosen so that axis z is orthogonal to the interface on which crack $|x| < l$ is located, and a wave is incident to the interface at angle θ. During propagation of an SH wave in the Oxz plane, the displacement vector has only one nonzero component u_y. The harmonic factor $\exp(-i\omega t)$ is from this point on omitted (ω, circular frequency; t, time).

Elastic Lamé constants λ^j, μ^j, density of materials ρ^j, and wave fields are designated by upper index $j = 1$ for the upper half-space, and $j = 2$ for the lower. The harmonic oscillations in each of the isotropic half-spaces are determined by the equations written with respect to displacements:

$$\mu^j\left(\frac{\partial^2 u_y^j}{\partial x^2} + \frac{\partial^2 u_y^j}{\partial z^2}\right) = \rho^j \omega^2 u_y^j, \quad j = 1, 2. \qquad (1)$$

The nonzero components of the stress tensor, according to the Hooke law, are expressed through displacements as follows:

$$\sigma_{yx}^j = \mu^j \frac{\partial u_y^j}{\partial x}, \quad \sigma_{yz}^j = \mu^j \frac{\partial u_y^j}{\partial z}.$$

For definiteness, we set it such that the plane wave is incident from the bottom half-space $z < 0$; as well, it partially passes, and is partially reflected by, the interface and a crack. In other words, a full wave field is the superposition of field $y_y^{j,\,\text{in}}$ without defect and the field scattered by the crack $u_y^{j,\,\text{sc}}$. The field without defect

$$u_y^{\text{in}} = \begin{cases} u_y^{1,\,\text{in}} = e^{ik_4^1(z\cos\theta + x\sin\theta)} + R^- e^{ik_4^1(-z\cos\theta + x\sin\theta)}, & z < 0, \\ u_y^{2,\,\text{in}} = T^- e^{ik_4^2(z\cos\theta_1 + x\sin\theta_1)}, & z > 0 \end{cases} \qquad (2)$$

is written via the amplitude transmissivity and reflection coefficients

$$R^- = \frac{\mu^1 k_4^1\cos\theta - \mu^2 k_4^2 \cos\theta_1}{\mu^1 k_4^1 \cos\theta + \mu^2 k_4^2\cos\theta_1},$$

$$T^- = \frac{2\mu^1 k_4^1 \cos\theta}{\mu^1 k_4^1\cos\theta + \mu^2 k_4^2 \cos\theta_1}.$$

Here $k_m^j = \omega/s_m^j$, $m = 1, 4$ are wave numbers, $k_4^1 \sin\theta = k_4^2 \sin\theta_1$, and s_4^j is the velocity of S waves (s_1^j is the velocity of R waves).

Scattered and incident fields satisfy the equation of motion (2) and the boundary conditions at the interface:

$$\begin{cases} u_y^{1,\,\text{sc}} = u_y^{2,\,\text{sc}}, & |x| > l, \\ \sigma_{yz}^{1,\,\text{sc}} = \sigma_{yz}^{2,\,\text{sc}}, & |x| > l, \\ \sigma_{yz}^{1,\,\text{sc}} = \sigma_{yz}^{2,\,\text{sc}}, & |x| < l. \end{cases}$$

The field scattered by a crack is written in the form of Fourier integrals

$$y_y^{\text{sc}} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} K^1(\alpha, z) L(\alpha) V(\alpha) \exp(-i\alpha x) d\alpha, & z < 0, \\ \displaystyle\int_{-\infty}^{\infty} K^2(\alpha, z) L(\alpha) V(\alpha) \exp(-i\alpha x) d\alpha, & z > 0 \end{cases} \qquad (3)$$

from the Fourier symbols of Green matrices of half-spaces $K^j(\alpha, z)$ and Fourier transforms $V(\alpha)$ of the function introduced for opening crack faces (see [23] for more detail)

$$v_y(x) = u_y^{1,\,\text{sc}}(x, 0^-) - u_y^{2,\,\text{sc}}(x, 0^+).$$

Substitution of the integral representations into the boundary conditions leads to the equation

$$\frac{1}{2\pi}\int_\Gamma L(\alpha) V(\alpha) \exp(i\alpha x) d\alpha = -i\mu^1 k_4^1 \cos\theta(1 - R^-), \quad |x| < l \qquad (4)$$

with the kernel

$$L(\alpha) = \frac{\mu^2 \sigma^1 \sigma^2}{\mu^1\sigma^1 + \mu^2 \sigma^2},$$

$$\sigma^j = \sqrt{(k_4^j)^2 - \alpha^2}, \quad \operatorname{Im}\sigma^j \ge 0, \quad \operatorname{Re}\sigma^j \le 0.$$

An unknown displacement jump $v_y(x)$ is expanded in a series,

$$v_y(x) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x/l), \qquad (5)$$

by Chebyshev second-order polynomials,

$$\psi_n(s) = \frac{\sin(n \arccos s)}{\sin s},$$

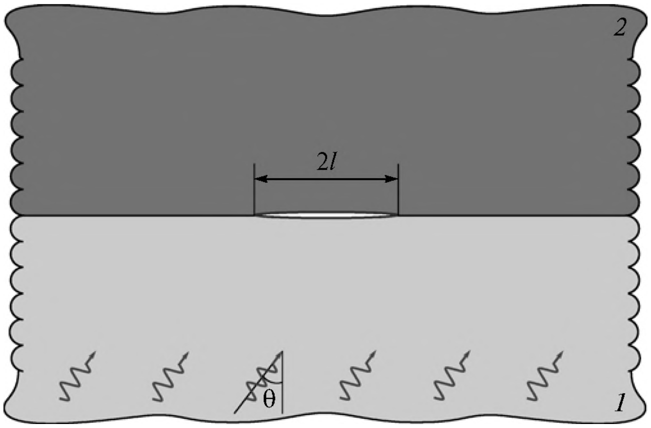

Fig. 1. Geometry of the single-crack problem.

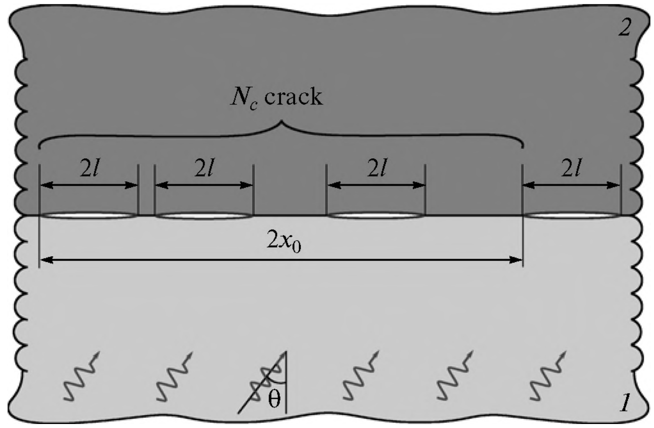

Fig. 2. Geometry of the problem for crack distribution.

which comprise the complete set of functions at the interval [−1, 1] and account for behavior of the solution at edges [24]. Substitution of expansion (5) into integral equation (4) and subsequent projection on ψ_n gives an infinite algebraic system of the equations:

$$\sum_{n'=1}^{\infty} Q_{nn'}\alpha_{n'} = -\frac{lc_{44}^2k_4^1k_4^2\cos\theta\sin\theta_1}{c_{44}^1k_4^1\cos\theta + c_{44}^2k_4^2\cos\theta_1}\Psi_n(k_4^1\sin\theta l),$$

where $\Psi_n(\alpha l) = \pi l i^n J_n(\alpha l)/\alpha$ is the Fourier transform from ψ_n with transform parameter α, and the coefficients in the left-hand side are Fourier integrals:

$$Q_{nn'} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c_{44}^2\sigma^1\sigma^2}{c_{44}^1\sigma^1 + c_{44}^2\sigma^2}\Psi_n(\alpha l)\Psi_{n'}(\alpha l)\alpha d\alpha.$$

Owing to the assumption of relative smallness $k_4^1 l$ for the Fourier symbol of the kernel of integral equation (4), it is possible to extract the linear part,

$$L(\alpha) = i\beta\alpha + O(1/\alpha), \quad \beta = \frac{\mu^2}{\mu^2 + \mu^1},$$

from a sufficiently small nonlinear component. Neglecting the latter, it is easy to obtain an asymptotic solution,

$$V(\alpha) = \frac{ik_4^1 lJ_1(\alpha l)}{2\alpha\beta}(1 - R^-),$$

or after application of an inverse Fourier transform,

$$v_y^0(x) = 2K\sqrt{l^2 - x^2},$$
$$K = i\frac{(\mu^1 + \mu^2)\cos\theta\cos\theta_1}{\mu^1k_4^1\cos\theta + \mu^2k_4^2\cos\theta_1}k_4^2k_4^1. \quad (6)$$

3.2. Distribution of Interface Cracks

As the next step, we consider the problem where instead of one crack with a width $2l$, there is an infinite set of interface cracks with width $2l$ (see Fig. 2). It is supposed that defects are distributed randomly and interaction between cracks can be neglected. In addition, it is possible to generalize it to the case of cracks of various size; however, this only complicates the analysis without leading to substantial corrections in solution [19] and the transmission coefficient is only necessary to derive SBCs.

An incident wave field free of defects, as it was done earlier, is determined by Eq. (2), and an exact expression cannot be obtained for a total wave field $u_y = u_y^{\text{in}} + u_y^{\text{sc}}$ owing to randomness of cracks. Instead of an exact solution, which, generally speaking, is not of much interest, ensemble average of the scattered field determined far from the half-space boundary is calculated. To determine ensemble average of a scattered field, the Betty-Rayleigh reciprocal theorem is suitable, applied to u_y^{sc} and $\tilde{u}_y^{\text{in}}$ obtained from u^{in} by replacement of $\theta := -\theta$:

$$\int_{S^-}(\tilde{u}_y^{\text{in}}\sigma_{yj}^{\text{sc}} - u_y^{\text{sc}}\tilde{\sigma}_{yj}^{\text{in}})n_j dS = 0,$$

where n_j is the normal external to a contour. The rectangular contour S^- relies on points $x = \pm x_0$, $z = -z_0$, and $z = 0^-$, and values x_0 and z_0 are chosen at random. Similarly, the Betty-Rayleigh theorem is applied to u_y^{sc}, $\tilde{u}_y^{\text{in}}$ for contour S^+, symmetric to S^- relative to axis $z = 0$. If we combine the obtained identities and apply them to the sum of averaging over ensemble of cracks, i.e., averaging of the field over all possible crack positions [21], then as a result, integrals along vertical segments $x = \pm x_0$ will not make any contribution and only integrals along horizontal segments remain:

$$\left(\int\limits_{z=z_0} - \int\limits_{z=-z_0}\right)(\tilde{u}_y^{\text{in}}\langle\sigma_{yz}^{\text{sc}}\rangle - \langle u_y^{\text{sc}}\rangle\tilde{\sigma}_{yz}^{\text{in}})dx + \left\langle\int\limits_D v_y\tilde{\sigma}_{yz}^{\text{in}}dx\right\rangle = 0. \quad (7)$$

Here, D is the damaged part of the boundary between half-spaces, and angle brackets denote the mean value over the ensemble of cracks. As well, the scattered wave field averaged over the ensemble represents plane waves extending from the interface in directions $\pm z$:

$$\langle u_y^{\text{sc}}\rangle = \begin{cases} P^- e^{ik_4^1(z\cos\theta + x\sin\theta)}, & z < 0, \\ P^+ e^{ik_4^2(z\cos\theta_1 + x\sin\theta_1)}, & z > 0. \end{cases} \quad (8)$$

The first term in (7) turns to zero, the second becomes simpler after substitution of expressions (2) and (8),

$$\int\limits_{z=-z_0}(\tilde{u}_y^{\text{in}}\sigma_{yz}^{\text{sc}} - u_y^{\text{sc}}\tilde{\sigma}_{yz}^{\text{in}})dx = -2ik_4^1\cos\theta\mu^1 P^-(2x_0),$$

and the third summand

$$\left\langle\int\limits_D v_y\tilde{\sigma}_{yz}^{\text{in}}dx\right\rangle = 2ik_4^1\cos\theta\mu^1(1-R^-) \times (2x_0)C\bar{v}_y\Psi_n(k_4^1\sin\theta l)$$

is expressed via the mean value of the function of opening crack faces,

$$\bar{v}_y = \frac{1}{2l}\int\limits_{-l}^{l} v_y(x)dx.$$

Here, $C = N_c l/x_0$ is the ratio of total length N_c cracks at interval $[-x_0, x_0]$ to the length of the considered area, i.e., damage. For the amplitudes of the scattered field from (7) and the law of conservation of energy follows

$$P^- = -\frac{1}{2}(1-R^-)C\bar{v}_y, \quad P^+ = -\frac{1}{2}(1+R^-)C\bar{v}_y.$$

As a result, the full transmission coefficient

$$\tilde{T} = T^- + P^+ = T^-\left(1 - \frac{1}{2}C\bar{v}_y\right). \quad (9)$$

For $\bar{v}_y$, it is possible to use expansion (5), then according to the properties of orthogonal Chebyshev polynomials, the mean amplitude of crack opening $\bar{v}_y = \pi\alpha_1 l/2$. If we use asymptotics (6), the average value of the crack opening displacement for low frequencies has an evident and convenient form: $\bar{v}_y^0 = \pi K l$.

3.3. Spring Boundary Conditions

The final stage consists in deriving the dependence between stiffness in SBCs and the parameters of the damaged interface. With this aim, we calculate the coefficient of transmissivity of a plane wave through two half-spaces, the adhesion between which is established by SBCs with unknown stiffness. The latter is determined from the assumption on equality of the transmission coefficients of the distributed spring and crack distribution at the interface.

Propagation of a plane wave through the boundary of the half-spaces connected by the distributed spring can be described by a representation similar to (2). Here, the amplitude coefficients of transmissivity and reflection

$$\hat{R}^- = \frac{i\mu^1k_4^1\cos\theta\mu^2k_4^2\cos\theta_1 + \kappa_y(\mu^1k_4^1\cos\theta - \mu^2k_4^2\cos\theta_1)}{i\mu^1k_4^1\cos\theta_1\mu^2k_4^2\cos\theta_1 + \kappa_y(\mu^1k_4^1\cos\theta + \mu^2k_4^2\cos\theta_1)},$$
$$\hat{T}^- = \frac{2\kappa_y\mu^1k_4^1\cos\theta}{i\mu^1k_4^1\cos\theta\mu^2k_4^2\cos\theta_1 + \kappa_y(\mu^1k_4^1\cos\theta + \mu^2k_4^2\cos\theta_1)} \quad (10)$$

are determined from SBCs at the interface,

$$\sigma_{yz}^1 = \sigma_{yz}^2 = \kappa_y(u^1 - u^2).$$

In the transition to limit $\kappa_y \to \infty$, wave fields can be considered continuous, and coefficients $\hat{R}^-$ and $\hat{T}^-$ coincide with R^- and T^-; at $\kappa_y \to 0$, full reflection takes place. To determine the value of κ_y it suffices to equate (9) with (10); i.e., to make the distributed spring equivalent to the crack distribution. It is preliminarily necessary to transform coefficients to the general form, which is done by expansion into a series at degrees κ_y of transmission coefficient (10):

$$\hat{T}^- = T^-\left(1 - \frac{i\mu^1k_4^1\cos\theta\mu^2k_4^2\cos\theta_1}{\mu^1k_4^1\cos\theta + \mu^2k_4^2\cos\theta_1}\kappa_y^{-1}\right) + O(\kappa_y^{-2}).$$

Equating the latter and using (6), we can obtain the stiffness value of the distributed spring:

$$\kappa_y = \frac{4}{C\pi l}\frac{\mu^1\mu^2}{\mu^1+\mu^2}\left(\frac{2\Psi_1(k_4^1\sin\theta l)}{\pi l}\right)^2.$$

The latter expression can be simplified owing to the smallness of $k_4^1 l$ with application of asymptotics for Bessel functions $J_n(\alpha) \approx 2^{-n}\alpha^n/n!$ at small α:

$$\kappa_y = \frac{4}{C\pi l}\frac{\mu^1\mu^2}{\mu^1+\mu^2}. \quad (11)$$

The expression for stiffness (11) in SBCs coincides with the value obtained in [20] for a normal angle of incidence. Since κ_y does not actually depend on the angle of incidence, then SBCs can be applied to describe delamination of different dimensions in waveguides of any type (for example, layer, cylindrical structure, wedge, etc.). We should also note that for identical materials, stiffness value (11) is close to the one obtained at low frequencies by Achenbach and Lee in [15] from energy conservation law.

4. NORMAL κ_z AND SHEARING κ_x COMPONENTS (PLANE CASE)

Passing to determination of components κ_z and κ_x of the stiffness matrix is done similarly that considered above, but already in the statement of the plane problem (see Figs. 1, 2). The displacement vector in the plane case has two nonzero components $\mathbf{u}^j = \{u_x^j, u_z^j\}$ and satisfies the Lamé equations

$$c_{11}^j\nabla\nabla\cdot\mathbf{u}^j - \mu^j\nabla\times(\nabla\times\mathbf{u}^j) + \rho^j\omega^2\mathbf{u}^j = 0,$$
$$j = 1, 2.$$

For convenience, here we introduce $c_1^j = \lambda^j + 2\mu^j$ and $c_2^j = \mu^j$. Similar to the antiplane case, first we consider a plane P or SV wave incident on an interface with a crack; accordingly, index $s = 1, 2$ (Fig. 1). With incidence of an elastic plane wave at a random angle in each of the half-spaces, two types of waves are excited and the coefficients of transmissivity and reflection have a cumbersome form [25]. This would complicate the derivation of values for κ_z and κ_x; therefore, only a normal angle of incidence is considered at which there are no additional types of waves. A field without defect,

$$\mathbf{u}^{\text{in}} = \begin{cases} \mathbf{p}^s(e^{ik_s^1 z} + R_s^- e^{-ik_s^1 z}), & z < 0, \\ \mathbf{p}^s T_s^- e^{ik_s^2 z}, & z > 0, \end{cases} \quad (12)$$

is expressed via the amplitude coefficients of transmissivity and reflection,

$$R_s^- = \frac{c_s^1k_s^1 - c_s^2k_s^2}{c_s^1k_s^1 + c_s^2k_s^2}, \quad T_s^- = \frac{2c_s^1k_s^1}{c_s^1k_s^1 + c_s^2k_s^2},$$

and propagation vectors $\mathbf{p}^P = \{0, 1\}$ for the P wave and, correspondingly, $\mathbf{p}^S = \{1, 0\}$ for an SV wave. For displacement vector $\mathbf{u}^{\text{sc}}$ of the field scattered by a crack, an interval representation similar in form to (3) is justified (for more detail, see [23]).

In the plane case, we also apply the Galerkin method with expansion, in the general case, by Jacobian polynomials taking into account oscillation [24]. This oscillation is insignificant and can be ignored, having passed to simpler second-order Chebyshev polynomials:

$$\{v_x, v_z\} = \sum_{n=1}^{\infty}\boldsymbol{\alpha}_n\psi_n(x/l).$$

From the assumption of smallness $k_s^1 l$ with application of integral approach [23], an asymptotic solution can be obtained,

$$\begin{pmatrix} v_{0,x}^P \\ v_{0,z}^P \end{pmatrix} = iH_P\begin{pmatrix} i\beta_2 \\ \beta_1 \end{pmatrix}\sqrt{l^2 - x^2},$$

$$\begin{pmatrix} v_{0,x}^S \\ v_{0,z}^S \end{pmatrix} = iH_S\begin{pmatrix} \beta_1 \\ -i\beta_2 \end{pmatrix}\sqrt{l^2 - x^2},$$

$$\beta_1 = \frac{c_1^1}{(\lambda^1+\mu^1)\mu^1} + \frac{c_1^2}{(\lambda^2+\mu^2)\mu^2}, \quad (13)$$

$$\beta_2 = \frac{1}{\lambda^1+\mu^1} - \frac{1}{\lambda^2+\mu^2},$$

$$H_s = \frac{c_s^1c_s^2k_s^1k_s^2}{c_s^1k_s^1 + c_s^2k_s^2},$$

for the plane problem for an incident P and SV wave, respectively. Figure 3 shows the accuracy of asymptotics (13) for different combinations of elastic properties $c_{ij}^2 = Bc_{ij}^1$; $B = 1, 2, 3, 4$, and at identical Poisson coefficients $\nu^j = \lambda^j[2(\lambda^j + \mu^j]^{-1}$ $\nu^1 = \nu^2 = 0.3333$, we can see a sufficiently small deviation of the asymptotic solutions from an exact solution at small $k_1^1 l < 0.5$.

Similarly to the scheme applied to the antiplane problem, we pass to solving the problem on multiple interface cracks (Fig. 2). The full transmission coeffi-

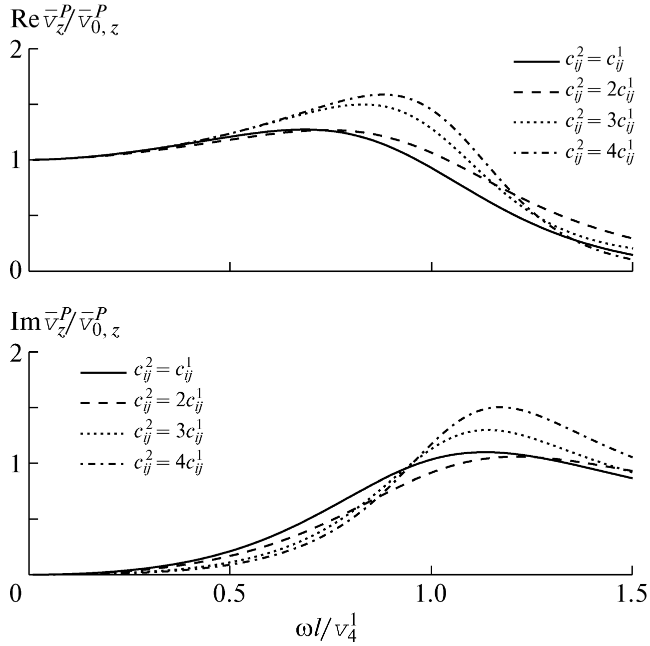

Fig. 3. Real and imaginary parts of the ratio $\bar{v}_z^P/\bar{v}_z^{0P}$ between mean values of exact $\bar{v}_z^P$ and asymptotic $\bar{v}_z^{0P}$ solutions, adopting purely imaginary values, see (13).

cient through the interface with the crack distribution is expressed as

$$\tilde{T}_s = T_s^- + P_s^+ = T_s^-\left(1 - \frac{1}{2}C\mathbf{p}^s \cdot \bar{\mathbf{v}}^s\right) \quad (14)$$

via the mean values of the corresponding components of displacement jumps on crack faces of corresponding problems; that is, it depends on the type of the incident P or SV waves.

The field of displacements of waves incident at the interface of two half-spaces connected by the distributed spring is similar to (12),

$$\mathbf{u} = \begin{cases} \mathbf{p}^s(e^{ik_1^1 z} + \hat{R}_s e^{-ik_s^1 z}), & z < 0, \\ \mathbf{p}^s \hat{T}_s e^{ik_s^2 z}, & z > 0, \end{cases}$$

and the transmissivity and reflection coefficients at $s = P, S$ are derived from SBCs:

$$\hat{R}_s^- = \frac{ic_s^1 k_s^1 c_s^2 k_s^2 + \kappa_s(c_s^1 k_s^1 - c_s^2 k_s^2)}{ic_s^1 k_s^1 c_s^2 k_s^2 + \kappa_s(c_s^1 k_s^1 + c_s^2 k_s^2)},$$

$$\hat{T}_s^- = \frac{2\kappa_s c_s^1 k_s^1}{ic_s^1 k_s^1 c_s^2 k_s^2 + \kappa_s(c_s^1 k_s^1 + c_s^2 k_s^2)}.$$

In order to take the final step in determining κ_s it is necessary to put $\hat{T}_s^-$ equal to transmission coefficient (14) for the problem on stochastic crack distribution [10]. The use of low-frequency asymptotics (13) makes it possible to obtain a complex-value expression for

$$\kappa_s = \frac{8}{\pi l C \beta_1} - iH_s,$$

which leads to energy losses in the system. However, at small values $k_s^1 l$, the material part prevails and, correspondingly, the constants for SBCs coincide independently of wave type:

$$\kappa_x = \kappa_z = \frac{8}{C\pi l\beta_1}.$$

5. CONCLUSIONS

We have considered an approach to studying nonperfect contact of interfaces based on application of SBCs. The main advantages of the spring model are boundary conditions for single delamination that are more general in comparison to the single-crack model (if $\kappa = 0$, a crack occurs). In addition, in models for describing defect concentration zones, it is possible to use SBCs on the damaged surface instead of multiple cracks. From the assumption on equivalence of the model with stochastic small-crack distribution and SBC models, we have derived the relations between the stiffness and delamination parameters. Derivation is carried out in the antiplane and plane statements of the problem, and the obtained stiffness values are similar: they include fracturing, the characteristic size of cracks, and elastic moduli. For the antiplane problem, a solution is obtained for any angle of incidence, which under our assumptions of smallness of cracks and wavelengths, hardly depends on the angle, which demonstrated the possibility of applying the model both for delaminations of finite dimensions and in limited waveguides. Thus, we have demonstrated the possibility of applying the developed approach in the two-dimensional case and have derived the relations for stiffnesses κ_i. The estimates obtained here for constants at the contact of different materials are in agreement with the constants obtained in [10] and [15] for identical materials.

The developed scheme can be expanded with some changes for describing nonperfect contact of materials in the three-dimensional case, including anisotropic materials (nondiagonal stiffness matrix κ). If microcracks have a clearly extended appearance (for example, along fibers in a composite), the stiffness values obtained in this study are applicable in this case.

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