ISSN 1063-7710, Acoustical Physics, 2010, Vol. 56, No. 6, pp. 871–876. © Pleiades Publishing, Ltd., 2010.

Travelling Waves in Microstructure as the Exact Solutions to the 6th Order Nonlinear Equation[1]

A. Casasso[a], F. Pastrone[a]*, and A. M. Samsonov[b]**

[a] *Department of Mathematics, University of Torino, Torino, 10123 Italy*

[b] *The Ioffe Physical Technical Institute of the Russian Academy of Sciences, St. Petersburg, 194021 Russia*

**e-mail: franco.pastrone@unito.it; **samsonov@math.ioffe.ru*

Received March 9, 2010

Abstract—The travelling wave solutions to the nonlinear partial differential equation of 6th order are obtained for a solid having two different spatial scales introduced in the microstructure. The slaving principle method is applied, and the exact explicit solution is found in terms of the doubly periodic Weierstrass elliptic function for the corresponding ODE. Several particular cases are discussed for various parameter values, e.g., the solitary "mexican hat" pulse is found with polarity, depending on microstructure parameters.

DOI: 10.1134/S1063771010060114

1. INTRODUCTION

Recently a considerable attention has been paid to the materials which exhibit a compound structure often called as "complex materials." The term has different meanings, referring to the theory of complexity, but in our context it means materials characterized by intrinsic spacial scales in the matter, such as the lattice period, the size of a crystalline or a grain, the distance between microcracks etc. In other words, all properties typical for materials used in modern technology, as for polycrystalline solids, ceramic composites, alloys, granular materials, may be described in corresponding spacial scales. Since 70-es the general theories introduced by Mindlin [1], Capriz [2], and Eringen [3] are widely used to underline an essence of dispersive effects due to the scale dependence in various physical models of wave propagation. However, as a rule the governing nonlinear equations contain spacial parameters, become quite awkward for analytical treatment and, consequently, require numerical simulations. For this reason any exact solution is expected to be of original interest.

The renewed attention nowadays to modeling of materials with different intrinsic spatial parameters is caused by nanotechnology. For example, dielectric materials supplied with nanoinclusions may become conductive due to enormously high surface-volume ratio of nanoparticles, 2–5% of nanofilaments may increase the stiffness of composites in times, etc.

Experiments in acoustical and mechanical behaviour of nanostructures *in situ* are still rare and very complicated, that calls for an importance of simulation of elastic features and deformation dynamics. Often the averaging parameters of micro- and nanocomposites are sufficient for detailed description of external impacts and wave propagation, see, e.g., [4, 5], and any direct relation between the wave parameters and elastic constants in micro- and nanoscale is of substantial interest.

We consider a microstructured model of a material and reduce the wave propagation problem in the complex material to the integration of the 6th order nonlinear partial differential equations (PDE). It is worth to note that many attempts were made to study various physical problems governed by the higher order nonlinear equations, and, since the gravity-capillary water waves modeling, the 5th order Korteweg–de Vries (KdV) equation remains among the most popular for description of a chain of coupled nonlinear oscillators, magneto-sound wave propagation in plasma etc. In fact, this equation is an evolutionary (unidirectional) version of the corresponding 6th order nonlinear quasi hyperbolic (bidirectional) equation, similar to the correspondence between the KdV and the Boussinesque equations. To name only a few, we mention the solitary wave solution of the 5th order KdV equation, obtained in numerical simulation for shallow water waves, that was found by a series solution in exponentials in [6]. Later in [7] that equation was integrated numerically under periodic initial and boundary conditions, while the homotopy analysis was applied to obtain the soliton solution to the same problem in [8].

In this paper we deal with a microstructured model, having two different spatial scale levels, e.g., the micro- and the nanoscale. We refer to the model developed in [9, 10], and in Section 2 the field equations will be obtained as the Euler–Lagrange equations of a suitable Lagrangean, as shown already in [10], assuming the absence of dissipation. A particular choice of the strain energy function, similar to that one used in the case of the only microstructure, will

[1] The article is published in the original.

allow us to deal with the explicit field equations. The usage of the slaving principle and the introduction of a phase variable both provide a reduction of the coupled governing equations to the one leading PDE, and to the 6th order ODE, resp. In Section 3 this ODE is reduced to the 4th order ODE, and, following an idea from [17], we find some exact solutions in terms of the Weierstrass elliptic function, that may be animated using *Mathematica*™.

Finally, in appropriate limiting cases we obtain, upon some additional relationships for the coefficients, the solitary pulse solutions, having typical "Mexican hat" shape. In Section 4 the physics of the results obtained above, of the parameter relations, and of the restrictions required is discussed.

2. STATEMENT OF THE PROBLEM

We consider an one-dimensional (1+1D) microstructured model with two different scale levels applied for the microstructure. Instead of the two-scale elastic system, containing both macro- and microstructures, we introduce a material, which is supposed to be a compound of an elastic macrostructure, a first level microstructure and a second level microstructure at much smaller scale. The last may be interpreted as a nanostructure, up to some extent (see [10, 12]).

Therefore, following the model, we deal with three different scalar functions: the one for the macrostructure and two for the microstructures, one for each scale level. The model of a material is the one-dimensional manifold, and we consider the material coordinates in space x and in time t; and the functions $v = v(x, t)$ for the macrostructure, $\varphi = \varphi(x, t)$ and $\psi = \psi(x, t)$, respectively, for the first and the second scale level in microscale. The macrosolids is supposed to be purely elastic, and both the first and second level microstructures satisfy the same generalized elasticity hypothesis as well, therefore the existence of an internal strain energy is assumed.

In general, the strain energy function in elastic solids with microstructures is assumed to be a function of the vector fields and their gradients [13]. Because of objectivity, we can write this function as a function of the scalar components only, namely,

$$W = W(v, v_x, \varphi, \varphi_x, \psi, \psi_x, x),$$

the kinetic energy K is a quadratic form in v_t, φ_t, ψ_t:

$$K = \frac{1}{2}(\rho v_t^2 + I_1\varphi_t^2 + I_2\psi_t^2),$$

where ρ is the one-dimensional mass density and remains equal to the constant density in the reference configuration, because we deal with the Lagrangian formulation, and the body is supposed to be homogeneous. The terms I_1 and I_2 are the inertia terms connected with the two different microstructure's scale levels. Moreover, we consider the models without dissipation, then the field equations will take the following form, as shown in [10]:

$$\begin{cases} \rho v_{tt} = \left(\dfrac{\partial W}{\partial v_x}\right)_x - \dfrac{\partial W}{\partial v} \\ I_1\varphi_{tt} = \left(\dfrac{\partial W}{\partial \varphi_x}\right)_x - \dfrac{\partial W}{\partial \varphi} \\ I_2\psi_{tt} = \left(\dfrac{\partial W}{\partial \psi_x}\right)_x - \dfrac{\partial W}{\partial \psi}, \end{cases} \quad (1)$$

where v is the displacement field, the subscripts denote derivatives with respect to time t, or to the spatial coordinate x, respectively.

A particular choice of the strain energy function W defines different nonlinear models, see [10]; in this paper we consider it in the following form:

$$\begin{aligned} W = {} & \frac{1}{2}\alpha v_x^2 + \frac{1}{3}\beta v_x^3 - A_1\varphi v_x + \frac{1}{2}B\varphi^2 \\ & + \frac{1}{2}C_1\varphi_x^2 - A_2\varphi_x\psi + \frac{1}{2}B_2\psi^2 + \frac{1}{2}C_2\psi_x^2. \end{aligned} \quad (2)$$

This function is the generalization of the strain energy function for nonlinear elastic solids with one microstructure level to our case, where the introduction of the cubic term v_x^3 represents the nonlinear behavior of the elastic matrix. The term appears in the framework of the so-called 5-constant or Murnaghan non-linear elasticity theory, widely used nowadays to take into consideration the nonlinearity of (macro)material, see, e.g., [7, 12, 17, 18]. The next terms in v_x can be added also to provide further, however, smaller corrections into macromodel of nonlinearly elastic solids. Elastic moduli data in the (next order) 9-constant nonlinear elasticity are still rare and often invalid.

The field equations from (1) can be written as:

$$\rho v_{tt} = \alpha v_{xx} + (\beta v_x^2)_x - A_1\varphi_x, \quad (3)$$

$$I_1\varphi_{tt} = C_1\varphi_{xx} + A_1 v_x - B_1\varphi - A_2\psi_x, \quad (4)$$

$$I_2\psi_{tt} = C_2\psi_{xx} + A_2\varphi_x - B_2\psi, \quad (5)$$

where α, β and A_i, B_i, C_i ($i = 1, 2$) denote material constants.

To obtain the governing equation in dimensionless form, it is necessary to introduce some parameters and constants, as follows:

$$C_1 = C_1^* l_1^2, \quad I_1 = \rho l_1^2 I_1^*, \quad A_1 = l_1 A_1^* \quad (6)$$

for the first level of microstructure, and:

$$C_2 = C_2^* l_2^2, \quad I_2 = \rho l_2^2 I_2^*, \quad A_2 = l_2 A_2^* \quad (7)$$

for the second scale level. The values l_1 and l_2 represent the size of the microstructural elements. Then we introduce two different parameters δ_i, $i = 1, 2$ characterizing the ratio between the microstructure, and the

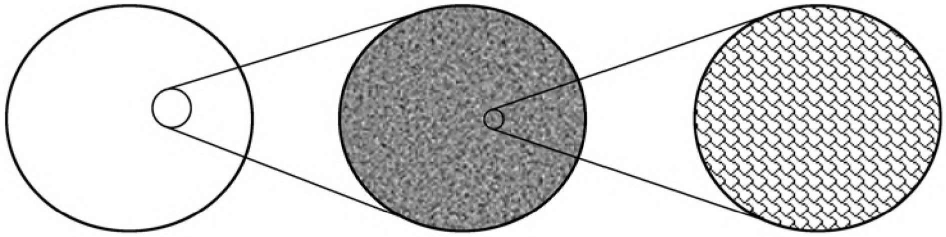

Fig. 1. A model with two levels of microstructure, e.g., micro- and nanolevels.

wave length L, and ϵ; accounting for small but finite elastic strain magnitude:

$$\delta_1 = \left(\frac{l_1}{L}\right)^2, \quad \delta_2 = \left(\frac{l_2}{L}\right)^2, \quad \epsilon = v_0 \ll 1, \tag{8}$$

where v_0 the intensity of the initial excitation. Introducing the macrostrain[1] $\nu = v_x$, the dimensionless variables

$$u = \frac{\nu}{v_0}, \quad X = \frac{x}{L}, \quad T = \frac{c_0}{L}t$$

and substituting the parameters (6), (7) and (8) into the previous system, we obtain the following coupled dimensionless equations:

$$\begin{cases} u_{TT} = \dfrac{\alpha}{\rho c_0^2}u_{xx} + \dfrac{\beta\epsilon}{\rho c_0^2}(u^2)_{XX} - \dfrac{A_1^* l_1}{\epsilon\rho c_0^2}\varphi_{XX} \\ \varphi = \dfrac{A_1^* l_1 v_0}{B_1}u - \dfrac{A_2^*\sqrt{\delta_2}}{B_1}\psi_X \\ + \dfrac{\delta_1}{B_1}[C_1^*\varphi_{XX} - \rho I_1^* c_0^2\varphi_{TT}] \\ \psi = \dfrac{A_2^*\sqrt{\delta_2}}{B_2}\varphi_X + \dfrac{\delta_2}{B_2}[C_2^*\psi_{XX} - \rho I_2^* c_0^2\psi_{TT}]. \end{cases} \tag{9}$$

The slaving principle [12] can now be used for further transformations. This procedure allows us to write one function in terms of the other; on this way we can obtain the governing equation for the function $u(x, t)$ only. To this end, we determine the variable ψ in terms of φ and its derivatives from $(9)_3$. Then the Eq. $(9)_3$ can be used to express φ in terms of derivatives of u. This expression will eventually be substituted into Eq. $(9)_1$ to obtain the one differential equation for u.

[1] The term "strain" is used here and further for brevity only; in fact, it is the longitudinal displacement gradient component, while expressions for genuine strains are nonlinear with respect to ν.

The resulting equation can be written as:

$$\begin{aligned} &u_{TT} + \alpha_1 u_{XX} + \alpha_2(u^2)_{XX} + (\alpha_3 u_{XX} + \alpha_4 u_{TT})_{XX} \\ &+ (\alpha_5 u_{4X} + \alpha_6 u_{TTXX} + \alpha_7 u_{4T})_{XX} = 0, \end{aligned} \tag{10}$$

where the following notation was introduced:

$$\alpha_1 = -\frac{\alpha B_1 - A_1^{*2} l_1^2}{B_1\rho c_0^2}; \quad \alpha_2 = -\frac{\varepsilon\beta}{\rho c_0^2};$$

$$\alpha_3 = \frac{A_1^{*2} l_1^2(\delta_1 C_1^* B_2 - \delta_2 A_2^*)}{B_1\rho c_0^2}; \quad \alpha_4 = -\frac{A_1^{*2} l_1^2\delta_1 I_1^*}{B_1^2};$$

$$\alpha_5 = \frac{A_1^{*2} l_1^2}{B_1^3 B_2^3\rho c_0^2}(\delta_1^2 C_1^{*2} B_2^3 - 2\delta_1\delta_2 A_2^{*2} B_2^2 C_1^*$$

$$- \delta_2^2 B_1 B_2 A_2^{*2} C_2^* - \delta_2^2 B_1 A_2^{*4});$$

$$\alpha_6 = \frac{A_1^{*2} l_1^2}{B_1^3 B_2^2}$$

$$\times(-2\delta_1^2 C_1^* B_2^2 I_1^* + \delta_1\delta_2 A_2^{*2} B_2 I_1^* + \delta_2^2 B_1 A_2^{*2} I_2^*);$$

$$\alpha_7 = \frac{\rho c_0^2 A_1^{*2}}{B_1^3}\delta_1^3;$$

$$u_{4X} \equiv u_{XXXX}; \quad u_{4T} \equiv u_{TTTT}.$$

The Eq. (10) above may be considered as the hierarchical equation in terms of u, where two different levels of microstructure are expressed in five different dispersive terms, and the higher order terms contain the parameters of the second level of microstructure.

We have obtained the 6th order PDE that is hardly to be solved explicitly in general case. However, we will find some exact travelling wave solutions of the PDE (10), when the equation can be reformulated in terms of the phase variable $z = x \pm Vt$ in the corresponding ODE, as follows:

$$\begin{aligned} &(V^2 + \alpha_1)u^{(\mathrm{II})} + \alpha_2(u^2)^{(\mathrm{II})} + (\alpha_3 + V^2\alpha_4)u^{(\mathrm{IV})} \\ &+ (\alpha_5 + V^2\alpha_6 + V^4\alpha_7)u^{(\mathrm{VI})} = 0, \end{aligned} \tag{11}$$

where V is the velocity of wave propagation.

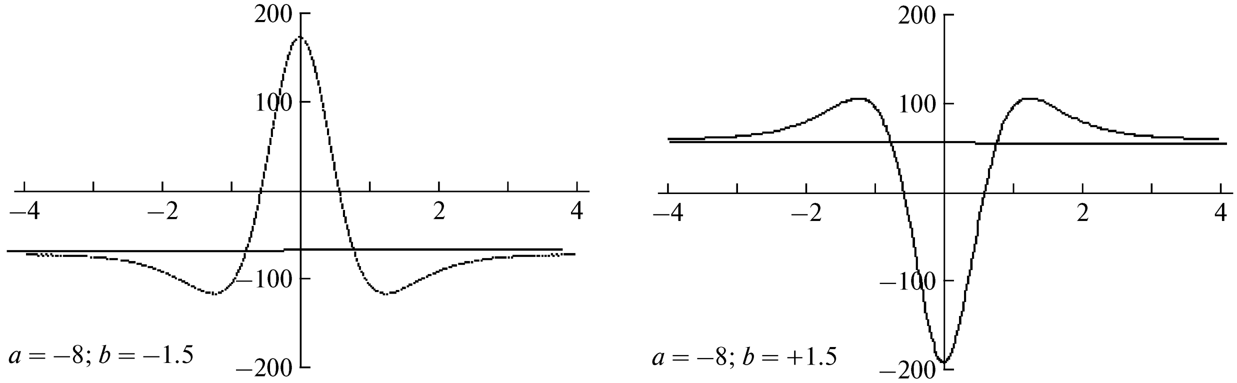

Fig. 2. The polarity of exact solitary wave solution to the 6th order ODE in the form of the "mexican hat" depends on model parameters.

3. EXACT SOLUTIONS TO THE 6th ORDER NONLINEAR ODE

Upon the introduction of z and integration twice with corresponding conditions at infinity $|z| \to \infty \Rightarrow u, u' \to 0$ the Eq. (11) may be rewritten as the nonlinear ODE of the 4th order:

$$u^{(IV)} + au^{(II)} + bu^2 + cu = 0, \quad (12)$$

where obviously:

$$a = (\alpha_3 + V^2\alpha_4)/\chi; \quad b = \alpha_2/\chi;$$
$$c = (\alpha_1 + V^2)/\chi; \quad \chi = \alpha_5 + V^2\alpha_6 + V^4\alpha_7. \quad (13)$$

Following the method described in [17] and applied in [11], the exact solution to the ODE (12) in terms of elliptic functions, containing only poles as the critical singularities, can be found in the following form:

$$u = M\wp^2(z; g_2, g_3) + S\wp(z; g_2, g_3) + K, \quad (14)$$

where the coefficients K, M, S and invariants g_i of the Weierstrass elliptic function $\wp$ are defined as follows:

$$K = \frac{1679a^2 - 11661c + 6\sqrt{161\Delta}}{23322b}; \quad \Delta = 28561c^2 - 1296a^4; \quad (15)$$

$$g_2 = \frac{a^2 \pm \sqrt{\Delta/161}}{2028}; \quad M = -840/b; \quad S = -140a/(13b); \quad (16)$$

$$g_3 = -\frac{961a^4 + 5334(a^2 \pm \sqrt{\Delta/161})a^2 - 257049c^2 + 1449(a^2 \pm \sqrt{\Delta/161})^2}{1860243840a}. \quad (17)$$

Note that for the cubic nonlinearity, say, f^3 in Eq. (10) the exact explicit solution f of (11) in terms of $\wp$ will be written even simpler: $f = \kappa_0 + \kappa_1\wp$, as shown in [17].

In general, any solution in terms of $\wp$ represents the doubly periodic function having poles (of 4th order in our case) as singular points. The following restriction on the wave and material parameters $\Delta = 0$ leads to the relation for parameters a, c of the initial problem:

$$\Delta = 0 \Rightarrow c = 36a^2/169; \quad K = -35a^2/1014b;$$
$$g_3 = a^3/474552; \quad g_2 = a^2/2028; \quad (18)$$

and to the simpler form of the solution:

$$u|_{\Delta=0} = -\frac{35ba^2}{1014} - \frac{140}{13b}\wp\left(z; \frac{a^2}{2028}, \frac{a^3}{474552}\right)a$$
$$-\frac{840}{b}\wp^2\left(z; \frac{a^2}{2028}, \frac{a^3}{474552}\right), \quad (19)$$

which may be animated in *Mathematica*™ [15].

Usually the graph looks like two sharp axisymmetrical peaks near $z = \pm 8.8$ for $a \simeq 620$.

The function $\wp$ may be reduced to different harmonic or elliptic limiting representations in dependence of parameters, see, [16]. In the appropriate limit the Weierstrass elliptic function is reducible to the elliptic Jacobi *cn*-function and, further, to the bounded solution u_0 in terms of $\cosh^{-2}$ function, i.e., to *the solitary wave solution*, as follows:

$$u_0 = s\cosh^{-4}(z) + q\cosh^{-2}(z) + p; \tag{20}$$

$$p = -c/b = -\frac{-18928 + 3640a - 31a^2}{507b};$$
$$q = \frac{140(52+a)}{13b}; \quad s = -840/b, \tag{21}$$

which is valid under an additional restriction to the equation parameters

$$c = (-18928 + 3640a - 31a^2)/507 \tag{22}$$

and has the well known shape of the so called "mexican hat":

The influence of values of a is smaller than of the parameter b, on which value depend both a sign of the soliton and its amplitude.

Animation of the solitary wave solution with parameters defined in (20) is instructive for visual demonstration of the soliton *polarity changes* resulted from the variation of parameters a, b and available in *Mathematica*™ via the command,

$$Animate[Plot[p + q\cosh(z)^{-2} + s\cosh^{-4}(x), \{x, -4, 4\},$$
$$PlotRange \to \{-200, 200\}],$$
$$\{a, -10, 10\}, \{b, -10, 10\}]].$$

The approach used to obtain these solutions is similar to that introduced and grounded in [17], and can be applied to explicitly solve some other higher order ODEs, e.g., the 5th order KdV and the 5th order mKdV equations.

4. CONCLUSIVE REMARKS AND DISCUSSION

The effect of small heterogeneities on the overall behaviour of a material depends on morphological material characteristics such as shape, size and spatiotemporal distribution of various microstructural properties.

Sharp grain boundaries appear in many organic, polymeric and metallic compounds and ceramics. The surface free energy increases due to distinctive grain (or inclusion) boundaries, that leads to abnormal sensitivity of the bulky structure to local stress growth, corrosion and fracture. Modeling of microstructure is, for that reasons, of great interest and importance for detailed description of microfracture mechanism in modern materials caused by local energy concentration in solitary strain waves.

The importance of correct estimations and averaging in the elastic nonlinearity description was emphasized already in recent review papers, e.g., in [19, 20]. Variations in nonlinear acoustical and mechanical properties of microstructured solids are very high as a rule, and one of the reasons consists in presence of very small linearly elastic components with quite different moduli, e.g., cracks or small inclusions of soft matter, [21].

Despite of many numerical simulations and modeling on physical level of accuracy tentative exact solutions to the highly nonlinear problems of microstructure dynamics are of considerable interest even in 1+1D statement, see, e.g., [22, 23].

For example, the condition (22), being transformed in terms of the initial parameters of the model, has a form of the 4th order algebraic equation for the velocity V^2. The general explicit solutions to it may be written, however, they are quite cumbersome. In particular, for $\alpha_7 \to 0$ and the material constants $B_1 = 0$; $A_2 = 0$ the equations for ψ and for v will be separated in (3), the equation for ϕ can be solved with a given function v in the right hand side, and from the condition (22) one has an estimation for the wave velocity in the form

$$V^2 = C_1/(2I_1\rho c^2), \tag{23}$$

which does not depend on $A_1 \neq 0$, neither on small δ_1, δ_2 in this case.

Another velocity value following from the condition (22) for $B_1 = 0$; $C_2 = 0$ and written in terms of the initial parameters of the model, has the form:

$$V^2 = \frac{C_1}{I_1\rho c^2}\left(1 - \frac{B_2C_1\delta_1 + A_2^2\delta_2}{2B_2C_1\delta_1 - A_2^2\delta_2}\right). \tag{24}$$

This case is less restrictive from the view point of the general model: in (3) the field equations for all unknowns v, ϕ, ψ remain coupled.

It is necessary to note, that nonlinear equations of higher order were already discussed in connection with acoustical problems [24, 25]. We hope, our results are also informative for physical applications and could hardly be found without explicit formulae for the travelling wave solutions.

REFERENCES

1. R. Mindlin, Arch. Rat. Mech. Anal. **16**, 51 (1964).
2. G. Capriz, *Continua with Microstructure* (Springer, 1989).
3. A. C. Eringen, *Microcontinuum Field Theories. Foundations and Solids* (Springer, 1999).
4. R. Tenne et al., Topics Appl. Phys. **111**, 635 (2007).
5. A. L. Ivanovsky, J. Inorg. Chem. **53**, 1368 (2008).
6. A. Sachs and J. H. Lee, Nuovo Cim. B **111**, 1429 (1996).
7. A. Salupere, J. Engelbrecht, O. Ilison, and L. Ilison, Math. Comp. Simul. **69**, 502 (2005).
8. S. Abbasbandy and F. Samadian Zakaria, Nonlinear Dyn. **51**, 83 (2008).

9. A. Casasso, PhD Thesis (Univ. di Torino, 2009).
10. A. Casasso and F. Pastrone, Wave Motion (2010), doi: 10.1016/j.wavemoti.2009.12.006.
11. A. Casasso, F. Pastrone, and A. M. Samsonov, Proc. Estonian Acad. Sci. Phys. Math. **56**, 75 (2007).
12. J. Engelbrecht, F. Pastrone, M. Braun, and A. Berezovski, in *Universality of Nonclassical Nonlinearity*, Ed. by P. P. Delsanto (Springer, 2007), pp. 29–48.
13. J. L. Ericksen, Int. J. Solids Struct., 371 (1970).
14. F. Pastrone, Math. Mech. Solids **10**, 349 (2005).
15. *Mathematica* and *Wolfram Mathematica* are trademarks of Wolfram Research, Inc.; www.wolfram.com.
16. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (US Nat. Bureau of Standards, 1964).
17. A. M. Samsonov, *Strain Solitons in Solids and How to Construct Them* (Chapman and Hall, CRC, 2001).
18. L. K. Zarembo and V. A. Krasil'nikov, Sov. Phys. Usp. **102**, 549 (1970).
19. O. V. Rudenko, A. I. Korobov, and M. Yu. Izosimova, Acoust. Phys. **56**, 151 (2010).
20. L. A. Ostrovsky and O. V. Rudenko, Acoust. Phys. **55**, 715 (2009).
21. V. Yu. Zaitsev, V. E. Nazarov, and V. I. Talanov, Phys. Usp. **176**, 97 (2006).
22. E. Majewski, in *Earthquake Source Asymmetry, Structural Media and Rotation Effects*, Ed. by R. Teisseyre, E. Majewski, and M. Takeo (Springer, 2006), pp. 296–300.
23. A. Berezovski, J. Engelbrecht, and G. Maugin, in *Mechanics of Microstructured Solids*, Lect. Notes Appl. Comput. Mech., vol. 46 (Springer, 2009), pp. 21–28.
24. O. V. Rudenko and V. A. Robsman, Dokl. Phys. **47** (6), 443 (2002).
25. M. V. Averiyanov, M. S. Basova, and V. A. Khokhlova, Acoust. Phys. **51** (5), 495 (2005).