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Analysis of Rotational Motion of Material Microstructure Particles by Equations of the Cosserat Elasticity Theory

O. V. Sadovskaya and V. M. Sadovskii

Institute of Computational Modeling, Russian Academy of Sciences, Siberian Branch, Krasnoyarsk, Akademgorodok 50, bld. 44, 660036 Russia

e-mail: sadov@icm.krasn.ru

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Abstract—Oscillatory processes in media with microstructure under the action of concentrated impulse and time-periodic perturbations are analyzed within the Cosserat elasticity theory. According to the results of computations, such media are characterized by a resonance frequency equal to the frequency of natural oscillations of particle rotational motion. This frequency is a phenomenological parameter of a material. It was established that the oscillatory rotation of particles changes for monotone rotational motion with increasing intensity of shear strains.

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INTRODUCTION

The mathematical model of the Cosserat continuum [1–3], which takes into account rotational degrees of freedom of microstructure particles of a material, is used to describe the stressed-strained state of composite, granular, powder, microfractured, and micropolar media. Current studies on simulation of nanoscale structures show that this model results from a limit transition in discrete molecular-dynamic models with an unbounded increase in a number of particles [4–6]. Therefore, in the near future the model will find wide application.

A fundamental difference between the Cosserat medium model and the models of classical elasticity theory is that the former includes two simultaneous mechanisms of perturbation transmission. Along with traditional longitudinal and transverse waves caused by translational motion, there exist specific waves occurring due to transmission of rotational motion in a continuum of particles (grains, blocks, or clusters) connected with one another by compliant elastic bonds. Rotational motion of particles explains, for instance, some qualitative features of propagation of surface [7 and 8] and nonlinear [9] waves in the Cosserat medium.

Analysis of the model equations shows [10] that such a medium is characterized by eigenfrequency of rotational motion determined only by elasticity of a material and inertial properties of microstructure particles and independent of dimensions of a sample under study and boundary conditions on its surface. Therefore, using the methods of acoustic tomography [11 and 12], reliable techniques for determination of the phenomenological parameters of a model can be developed. The same principle can apparently be used to develop effective methods of resonance probing materials with micro- and nanostructure. This study is aimed at simulation of resonance methods of perturbation of the Cosserat media that cause the acoustic resonance at the eigenfrequency of particle rotational motion.

ANALYSIS OF LINEAR THEORY EQUATIONS

Translational motion of particles in the Cosserat elastic medium is characterized by velocity vector v and independent rotations are described by angular velocity vector ω. Along with asymmetric stress tensor σ, an asymmetric couple stress tensor m is introduced. The complete system of equations comprises the equations of motion, kinematic relations, and generalized law of the linear elasticity theory:

$$\begin{gathered}\rho\dot{v} = \nabla\cdot\sigma + f, \quad j\dot{\omega} = \nabla\cdot m - 2\sigma^a + g,\\ \dot{\Lambda} = \nabla v + \omega, \quad \dot{M} = \nabla\omega,\\ \sigma = \lambda(\delta : \Lambda)\delta + 2\mu\Lambda^s + 2\alpha\Lambda^a,\\ m = \beta(\delta : M)\delta + 2\gamma M^s + 2\varepsilon M^a.\end{gathered} \tag{1}$$

Here, ρ is the density of a medium; j is the moment of inertia of particles in a unit volume; f and g are the vectors of body forces and moments, respectively; Λ and M are the strain and curvature tensors; δ is the metric tensor; and λ, μ, α, β, γ, and ε are the phenomenological elasticity coefficients for an isotropic material. The conventional notation and operations of the tensor analysis are used: a colon denotes double convolution; a dot over a symbol, a time derivative; and an asterisk, tensor transposition. Superscripts s and a

denote the symmetric and asymmetric components of the tensors:

$$\Lambda^s = \frac{\Lambda + \Lambda^*}{2}, \quad \Lambda^a = \frac{\Lambda - \Lambda^*}{2}.$$

Where needed, the asymmetric component is identified with the corresponding vector. In particular, the vector of tensor σ^a appears in the equations of motion.

The linear scale of the microstructure of a material is estimated using the formula $r = \sqrt{5j/(2\rho)}$ based on the model representation of a medium as close packing of an ensemble of spherical particles equal in radius.

In the componentwise form, system (1) written in velocities and stresses relative to the Cartesian coordinate system is

$$\begin{aligned} \rho \dot{v}_i &= \sigma_{1i,1} + \sigma_{2i,2} + \sigma_{3i,3} + f_i, \\ j\dot{\omega}_i &= m_{1i,1} + m_{2i,2} + m_{3i,3} + \sigma_{kl} - \sigma_{lk} + g_i, \\ a_1\dot{\sigma}_{ii} + a_2(\dot{\sigma}_{kk} + \dot{\sigma}_{ll}) &= v_{i,i}, \\ b_1\dot{m}_{ii} + b_2(\dot{m}_{kk} + \dot{m}_{ll}) &= \omega_{i,i}, \\ a_3\dot{\sigma}_{ik} + a_4\dot{\sigma}_{ki} &= v_{k,i} - \omega_l, \\ b_3\dot{m}_{ik} + b_4\dot{m}_{ki} &= \omega_{k,i}, \\ a_4\dot{\sigma}_{ik} + a_3\dot{\sigma}_{ki} &= v_{i,k} + \omega_l, \\ b_4\dot{m}_{ik} + b_3\dot{m}_{ki} &= \omega_{i,k}. \end{aligned} \qquad (2)$$

This form of the system is used for numerical implementation of the model. Here, the subscripts after a comma denote partial derivatives of space variables. For brevity, the following notations are used:

$$i, k, l = 1, 2, 3, \quad i \neq k \neq l,$$

$$k = i + 1 \bmod 3, \quad l = k + 1 \bmod 3,$$

$$a_1 = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}; \quad a_2 = -\frac{\lambda}{2\mu(3\lambda + 2\mu)};$$

$$a_3 = \frac{\mu + \alpha}{4\mu\alpha}; \quad a_4 = \frac{\alpha - \mu}{4\mu\alpha};$$

$$b_1 = \frac{\beta + \gamma}{\gamma(3\beta + 2\gamma)}; \quad b_2 = -\frac{\beta}{2\gamma(3\beta + 2\gamma)};$$

$$b_3 = \frac{\gamma + \varepsilon}{4\gamma\varepsilon}; \quad b_4 = \frac{\varepsilon - \gamma}{4\gamma\varepsilon}.$$

System (2) can be written in the matrix form [13 and 14]:

$$A\dot{U} = B^1 U_{,1} + B^2 U_{,2} + B^3 U_{,3} + QU + G, \qquad (3)$$

where U is the vector function comprising the components of the velocity and angular velocity vectors and the stress and couple-stress tensors:

$$U = (v_1, v_2, v_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{32}, \sigma_{31}, \sigma_{13}, \sigma_{12}, \sigma_{21}, \omega_1, \omega_2, \omega_3, m_{11}, m_{22}, m_{33}, m_{23}, m_{32}, m_{31}, m_{13}, m_{12}, m_{21}).$$

Matrix coefficients A, B^1, B^2, and B^3 are symmetric, Q is asymmetric, and G is the specified vector. Matrix A is positive definite under the conditions

$$3\lambda + 2\mu, \mu, \ \alpha > 0, \quad 3\beta + 2\gamma, \gamma, \ \varepsilon > 0,$$

that ensure positive potential energy of the elastic strain. In this case, system (3) is hyperbolic in the sense of Friedrichs. For hyperbolic systems, the Cauchy problem and the boundary problems with the dissipative boundary conditions are correct [15 and 16]. The characteristic properties of system (3) are described by the equation

$$\det(cA + n_1 B^1 + n_2 B^2 + n_3 B^3) = 0,$$

$$n_1^2 + n_2^2 + n_3^2 = 1,$$

the positive roots of which are velocities of the longitudinal waves (c_p), transverse waves (c_s), torsional waves (c_m), and rotational waves (c_ω):

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu + \alpha}{\rho}}, \quad c_m = \sqrt{\frac{\beta + 2\gamma}{j}}, \quad c_\omega = \sqrt{\frac{\gamma + \varepsilon}{j}}. \qquad (4)$$

In the one-dimensional case, when the sought functions depend only on time and one of the space variables x_1, system (2) divides into four independent subsystems describing the plane longitudinal waves,

$$\rho \dot{v}_1 = \sigma_{11,1}, \quad \dot{\sigma}_{11} = (\lambda + 2\mu) v_{1,1}, \quad \dot{\sigma}_{22} = \dot{\sigma}_{33} = \lambda v_{1,1}, \qquad (5)$$

the torsional waves,

$$j\dot{\omega}_1 = m_{11,1} + \sigma_{23} - \sigma_{32}, \quad \dot{\sigma}_{32} = -\dot{\sigma}_{23} = 2\alpha\omega_1, \quad \dot{m}_{11} = (\beta + 2\gamma)\omega_{1,1}, \text{ and } \dot{m}_{22} = \dot{m}_{33} = \beta\omega_{1,1}, \qquad (6)$$

and the transverse (shear) waves with rotation of particles,

$$\begin{aligned} \rho \dot{v}_2 &= \sigma_{12}, \quad \dot{\sigma}_{12,1} = (\mu + \alpha) v_{2,1} - 2\alpha\omega_3, \\ \dot{\sigma}_{21} &= (\mu - \alpha) v_{2,1} + 2\alpha\omega_3, \\ j\dot{\omega}_3 &= m_{13,1} + \sigma_{12} - \sigma_{21}, \\ \dot{m}_{13} &= (\gamma + \varepsilon)\omega_{3,1}, \quad \dot{m}_{31} = (\gamma - \varepsilon)\omega_{3,1}. \end{aligned} \qquad (7)$$

One more subsystem describing the transverse waves is obtained from (7) by changing the indices.

The general solution of subsystem (5) is expressed by d'Alembert's formula, according to which the longitudinal waves propagate with velocities $\pm c_p$, as in the

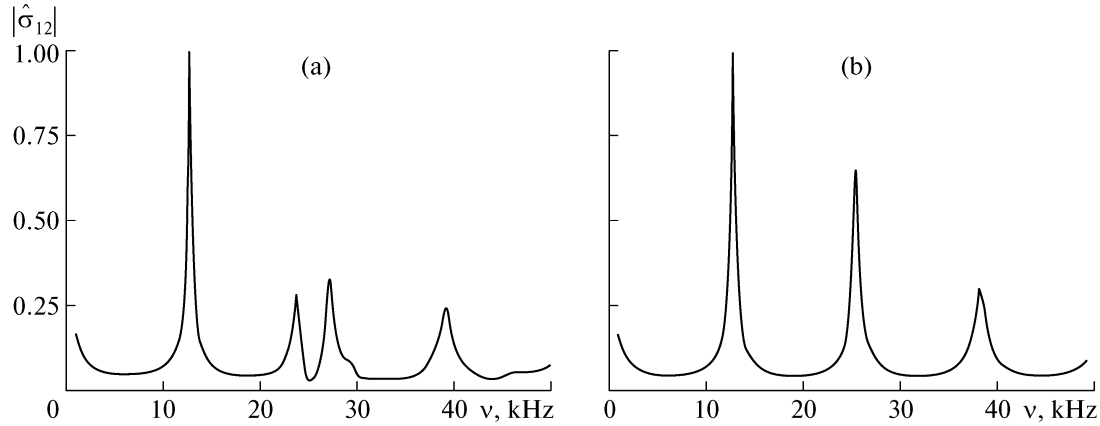

Fig. 1. Characteristic spectral curves of the tangential stress for the Cosserat continuum (a) and a momentless viscoelastic medium (b).

classical elasticity theory, and are not characterized by dispersion.

Subsystem (6) of the torsional waves is reduced to the telegraph equation relative to the angular velocity $\ddot{\omega}_1 = c_m^2 \omega_{1,11} - 4\alpha\omega_1/j$. The corresponding dispersion equation $c = \pi\nu c_m/\sqrt{\pi^2\nu^2 - \alpha/j}$, where c is the group velocity and ν is the cyclic frequency, determines a particular solution in the form of a monochromatic wave: $\omega_1 = C\exp(2\pi i\nu(t - x_1/c))$, where i is the imaginary unit and C is the complex constant. The telegraph equation solution independent of x_1 and obtained from here at $c \to \infty$ describes uniform self-oscillatory rotation of particles of the medium upon homogeneous shear with the period $T = \pi\sqrt{j/\alpha}$. The general solution of the telegraph equation is expressed as

$$\omega_1 = \int_0^{\xi} p'(\vartheta) J_0(-2\sqrt{(\xi-\vartheta)\eta})\,d\vartheta$$
$$+ \int_0^{\eta} q'(\vartheta) J_0(-2\sqrt{\xi(\eta-\vartheta)})\,d\vartheta$$
$$+ (p(0) + q(0)) J_0(-2\sqrt{\xi\eta}),$$

where $\xi, \eta = (t \pm x_1/c_m)\pi/T$, J_0 is the zero-order Bessel function of the first kind, and p and q are the arbitrary continuously differentiable functions that determine the Goursat conditions for the characteristics of the equation:

$$\omega_1(\xi, 0) = p(\xi) + q(0),$$
$$\omega_1(0, \eta) = p(0) + q(\eta).$$

The analysis of the solution shows that wavelike rotational motion of particles with the characteristic wavelength of about $c_m T$ is excited in the plane of the torsional wave fronts [14].

The results of numerical investigation of Eqs. (7) of the transverse waves on the basis of the Neumann–Richtmyer finite-difference scheme were reported in [13]. The computations were performed for different scales of the microstructure of a material in the one-dimensional problem on impulse action on an elastic medium of a periodic system of Λ-shaped impulses of tangential stress. The results showed that, in each fixed moment of time, the angular velocity and couple stresses are oscillating functions with the characteristic wavelength $c_s T$. Consequently, the occurrence of the oscillations, similar to the case of the torsional waves, is related to the rotational oscillations of particles: propagating along the medium, the transverse wave initiates such motion at the front plane.

On the basis of Eqs. (7), in the one-dimensional case the ways of resonance excitation were simulated by the expense of periodic boundary excitation with the frequency of natural oscillations of rotational motion of particles. Figure 1 depicts the tangential stress spectral curves obtained by numerical solution of the problem on uniform cyclic shear of a viscoelastic layer of finite thickness. The graphs correspond to the rigidly fixed bottom side of the layer. The solution describes also the torsional oscillations of a cylindrical sample with one edge rigidly fixed. In this case, the tangential stress depends linearly on radius and, hence, is proportional to σ_{12} and linear velocity is proportional to v_2. Figure 1a corresponds to the Cosserat medium; Figure 1b depicts the same curve for an ordinary momentless viscoelastic medium. The analogous graphs for perfect nonviscous media have a system of resonance peaks with infinite amplitudes. As it usually is, viscosity was used for smoothing. The shear process was described by Eqs. (7) where, according to the Boltzmann viscoelasticity theory, the products of the medium parameters on the kinematic characteristics of straining were replaced by the same characteristics using convolutions of the relaxation nuclei corresponding to this parameters. The boundary conditions of the problem were taken as

$$v_2|_{x_1=0} = \nu_0 \exp(2\pi i \nu t), \quad \omega_3|_{x_1=0} = 0, \qquad v_2|_{x_1=h} = \omega_3|_{x_1=h} = 0 \tag{8}$$

with the x_1 axis directed from the top down inward the layer (h is the layer thickness). The system was solved by a spectral method: after the Fourier transformation, the following system of amplitude equations was obtained:

$$4\pi^2\nu^2\rho\hat{v}_2 + (\hat{\mu}+\hat{\alpha})\hat{v}_{2,1} - 2\hat{\alpha}\hat{\omega}_{3,1} = 0, \qquad 4(\pi^2\nu^2 j - \hat{\alpha})\hat{\omega}_3 + (\hat{\gamma}+\hat{\varepsilon})\hat{\omega}_{3,11} + 2\hat{\alpha}\hat{v}_{2,1} = 0. \tag{9}$$

The solution of this system was built in the explicit form with regard for boundary conditions (8). The amplitude of the tangential stress was determined as

$$2\pi i\nu\hat{\sigma}_{12} = (\hat{\mu}+\hat{\alpha})\hat{v}_{2,1} - 2\hat{\alpha}\hat{\omega}_3.$$

The phenomenological parameters of a medium were selected by the experimental data for heavy oil in a rock at sufficiently low temperature where the material is in the solid state [17]. For this material, $\rho = 1114$ kg/m^3, $j = 0.01$ kg/m, $\mu = 966$, $\alpha = 52.2$ MPa, $\gamma + \varepsilon = 12.51$ N, and $h = 36.4$ mm. According to the Kelvin–Voigt theory used in the computations, the complex modules are linear functions of frequency; in particular, $\hat{\mu} = \mu + 2\pi i\nu\mu'$. The imaginary parts were chosen as to obtain the required smoothing of the solution.

Comparison of the graphs in Fig. 1 shows that in the Cosserat medium there is an additional resonance frequency of 23 kHz close to the frequency of rotational motion of particles and independent of layer thickness. This is confirmed by a great number of numerical experiments for different thicknesses. It appeared that the change in h leads to natural displacement of the periodic system of ground resonance frequencies which are approximately $\nu_k = kc_s/(2h)$, where $k = 1, 2, \ldots$, but the peak corresponding to the frequency $\nu_* = 1/T$ remains motionless.

The analogous computations for foamy polyurethane with the parameters $\rho = 340$ kg/m^3, $j = 4.4 \times 10^{-4}$ kg/m, $\lambda = 416$, $\mu = 104$, $\alpha = 4.33$ MPa, $\beta = -22.8$, $\gamma = 40$, and $\varepsilon = 5.3$ N [18] did not show a peak so noticeable against the background of the fundamental frequencies at the material resonance frequency of the same order of magnitude as that for heavy oil. This fact is explained by the substantially less pronounced couple properties expressed in the values of parameters j and α. It appears that in the materials with the low couple properties the resonance of rotational motion of particles can be excited, for example, by the periodic variation of the angular momentum at the layer boundary. Spectral curves for synthetic polyurethane are presented in Fig. 2. In

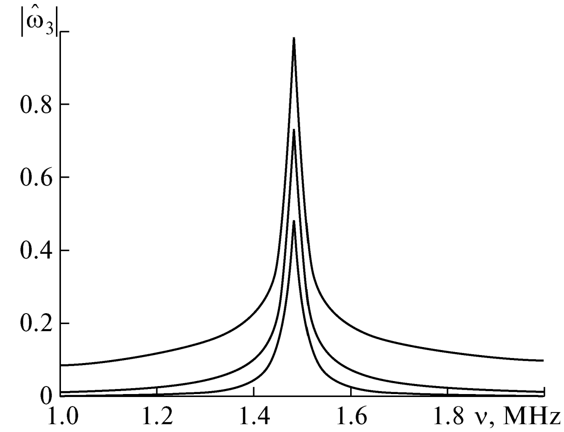

Fig. 2. Resonance peak of the angular velocity amplitude on the spectral curve for synthetic polyurethane.

this case, the parameters are $\rho = 590$ kg/m^3, $j = 5.31 \times 10^{-6}$ kg/m, $\lambda = 2.195$, $\mu = 1.033$, $\alpha = 0.115$ GPa, $\beta = -2.34$, $\gamma = 4.1$, and $\varepsilon = 0.13$ N. The results were obtained by the numerical computation of system (9) with the boundary conditions

$$\sigma_{12}|_{x_1=0} = 0, \quad m_{13}|_{x_1=0} = m_0 \exp(2\pi i\nu t), \qquad v_2|_{x_1=h} = \omega_3|_{x_1=h} = 0 \tag{10}$$

using the spectral-difference method.

The presented graphs correspond to $h = 10$ cm; however, they are nearly independent of layer thickness. The curves are related to different levels: the upper one corresponds to the layer boundary where the periodic perturbations are excited; the middle and lower ones, to the levels distant by a quarter- and half-thickness of the layer deep from the boundary. Analysis shows that the amplitude of the angular velocity has the only resonance peak at the frequency $\nu_* = 1.48$ MHz within the range of interest; the height of the peak is determined by the imaginary part of the parameter $\hat{\gamma} + \hat{\varepsilon}$ and drops with depth almost linearly.

Qualitatively similar results in the problem with boundary conditions (10) were given by the computations for the parameters of foamy polyurethane and heavy oil.

RESULTS OF SPATIAL COMPUTATIONS

The problems of numerical implementation of the Cosserat elastic medium in the geometrically linear approximation using multiprocessor computers were considered in [13 and 14]. The two-dimensional computations of the natural oscillations of particle rotational motion were presented in [13]. The results of numerical solution of the spatial Lamb's problem on momentary action of concentrated forces and moments at the half-space surface were reported in [14]. The computations were based on the existing

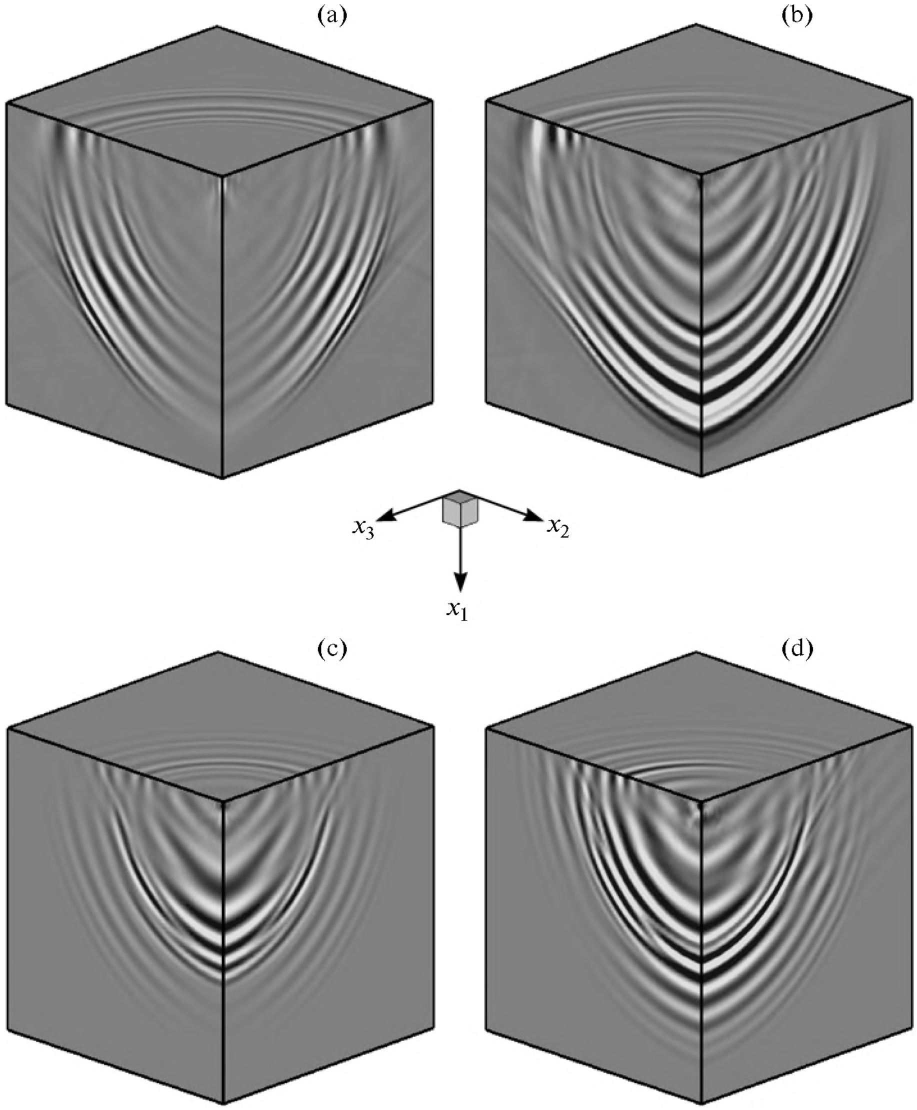

Fig. 3. Lamb's problem for different types of loading: level surfaces of stress σ_{11} under the normal loading (a), stress σ_{12} under the tangential loading (b), moment m_{11} under the action of the torsional moment (c), and moment m_{12} under the action of the rotational moment (d).

software developed by the authors. In the seismograms plotted using the results of computations in a SeisView system of geophysical data processing, loading waves of four types propagating with velocities (4) were identified: longitudinal, transverse, torsional, and rotational. The spatial computations confirmed that the main qualitative difference of the wave field in the Cosserat elastic medium from the classical elasticity theory is the occurrence of oscillations of particle rotational motion at the wave fronts. Comparative computations for different scales of the medium microstructure established the direct proportional dependence of the natural oscillation period on the scale.

Figure 3 depicts the level surfaces of stress and moments for Lamb's problem on the action of concentrated loading at the angular point of the upper boundary of the computation region. Loading of different types was considered. The level surfaces of stress σ_{11} under the normal loading $\sigma_{11} = -p_1\delta(x)\delta(t)$ are illustrated in Fig. 3a. The level surfaces of stress σ_{12} under the tangential loading $\sigma_{12} = -p_2\delta(x)\delta(t)$ are shown in Fig. 3b. The level surfaces of moment m_{11} under the action of the torsional moment $m_{11} = -q_1\delta(x)\delta(t)$ and those of moment m_{12} under the rotational moment $m_{12} = -q_2\delta(x)\delta(t)$ are presented in Figs. 3c and 3d, respectively. Behind the fronts of the transverse waves, one can clearly see the oscillations caused by rotational motion of particles of the material. The computational region with regard of the symmetry conditions is a quarter of half-space. In practice, the computations were performed on a cube with a side of 1 cm for

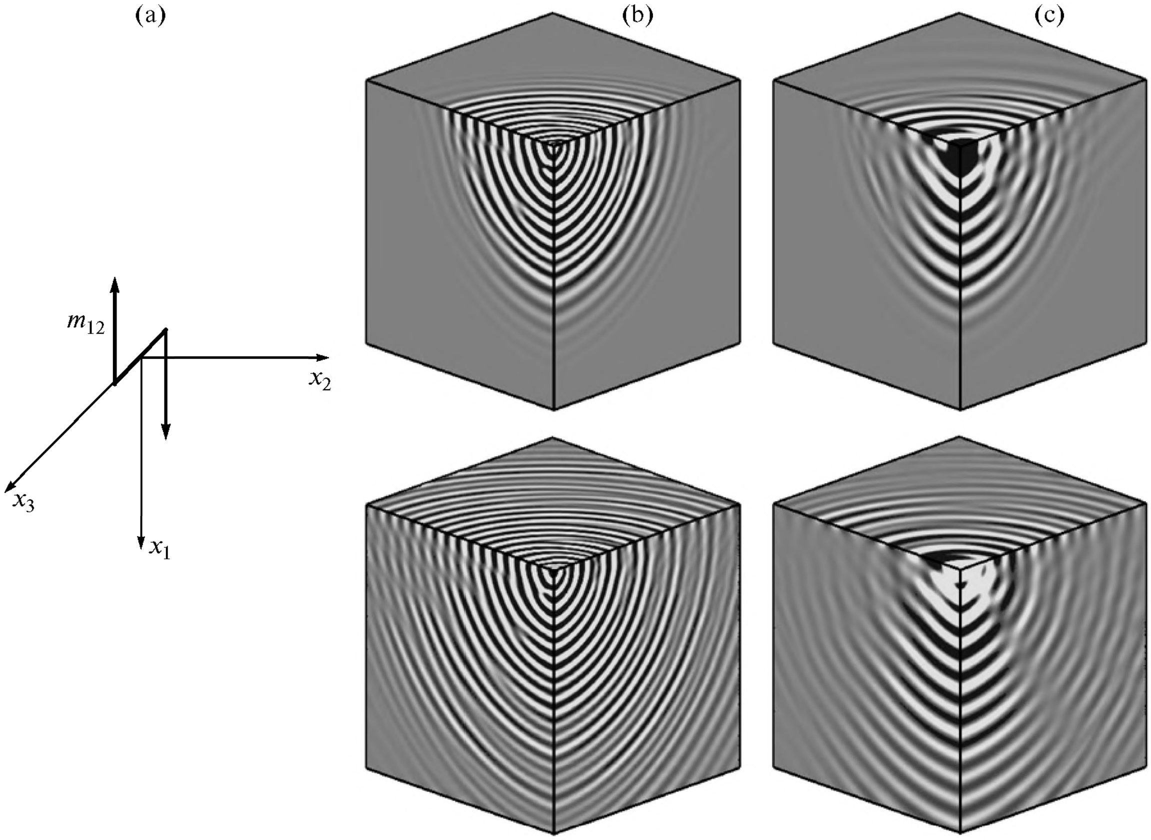

Fig. 4. Problem on the periodic action of the concentrated rotational moment: loading scheme (a) and the surfaces of the level of angular velocity ω_2 for the resonance (b) and nonresonance (c) frequencies in different moments of time (t = 6.5 and 13 μs on top and at the bottom, respectively).

the case of synthetic polyurethane. The figures correspond to the time moment 6.5 μs. To obtain acceptable accuracy, 64 cluster processors were required.

Below, we present the results of numerical solution of the problem on the action of the concentrated rotational moment $m_{12} = -q_2\delta(x)\sin(2\pi\nu t)$ periodically changing in time. Figure 4 presents the loading scheme for this problem (Fig. 4a) and the surfaces of the level of angular velocity ω_2 for the nonresonance frequency $\nu = 1.5\nu_*$ (Fig. 4b) and the resonance frequency $\nu = \nu_*$ (Fig. 4c) in different moments of time. The computations showed that, at external action frequency ν_* (Fig. 4c) equal to the eigenfrequency of rotational motion of particles, the amplitude grows with time and the oscillations smoothly damp upon moving off the point of loading application; such damping is characteristic of acoustic resonance. The analogous computations showed that the variants of specifying the time-periodic and spatially concentrated normal or tangential stress and the linear or angular velocity do not lead to noticeable resonance excitation of a medium.

NONLINEAR VERSION OF THE MODEL

In the geometrically nonlinear case, translational motion of a medium with the microstructure is described by the equation $x = x(\xi, t)$ relating the Lagrangian and Euler vectors of the centers of mass of particles. To take into account rotational motion, the orthogonal tensor $R = R(\xi, t)$ is introduced. The velocity vector is $v = \dot{x}$ and the angular velocity tensor is calculated as $\omega = \dot{R} \cdot R^*$. As a measure of strain, the tensor $\Lambda = \nabla_\xi x \cdot R$ is taken, which possesses the property that, upon motion of the medium as a rigid whole when the strain gradient tensor $(\nabla_\xi x)^*$ coincides with rotation tensor R, the former equals to the metric tensor, which corresponds to the strainless state of the material. In addition, the equation

$$\dot{\Lambda} \cdot R^* = \nabla_\xi v + \nabla_\xi x \cdot \omega \quad (11)$$

is valid, the linear approximation of which exactly corresponds to the kinematic equation for the tensor of strain velocities in the geometrically linear Cosserat model. Let $(\nabla_\xi x)^* = R_e \cdot V$ is the polar decomposition of the strain gradient tensor in the product of orthogonal R_e and symmetric V tensors. The construction of the tensor $\Lambda = V \cdot R_r$ takes into account both the normal medium strain described by the symmetric part of this decomposition and the relative particle rotation characterized by the tensor $R_r = R_e^* \cdot R$, while the tensor of particle rotation $R = R_e \cdot R_r$ is presented as superposition of the relative and translational rotations.

From the integral laws of conservation of momentum, angular momentum, and energy with regard of the laws of reversible thermodynamics, the system of differential equations of motion and the defining equations follows:

$$\rho_0 \dot{v} = \nabla_\xi \cdot \sigma + f,$$
$$\frac{\partial}{\partial t}(J \cdot \omega) = \nabla_\xi \cdot m + 2(\sigma^* \cdot \nabla_\xi x)^a + g, \qquad (12)$$
$$\sigma \cdot R = \frac{\partial \Phi}{\partial \Lambda}, \quad m = \frac{\partial \Phi}{\partial M}.$$

Here, ρ_0 is the initial density of a medium, J is the symmetric and positively defined tensor of inertia, and Φ is the internal energy per unit volume, i.e., the elastic stress potential depending on tensors Λ, M, and entropy S. For the adiabatic processes, the entropy is constant and enters system (12) as a parameter. In the nonlinear version of the model, the stress tensors are related to the initial configuration and curvature tensor M is governed by the equation $\dot{M} = \nabla_\xi \omega$, which includes the Lagrangian variable gradient.

Tensor of inertia J changes depending on time in accordance to the equation $J = R \cdot J^0 \cdot R^*$, which can be explained by the transition to the accompanying coordinate system associated with a rotating particle. After time differentiation, it yields the differential equation $\dot{J} = \omega \cdot J - J \cdot \omega$. System (11), (12) differs from the system of equations for the model of a micropolar medium [19] only in some details.

In the case of an isotropic medium, the tensor of inertia is spherical, $J^0 = j\delta$, and the internal energy depends only on the invariants of tensors Λ and M. As a complete system of independent invariants, three invariants from the symmetric part and one invariant from the antisymmetric part of each of the tensors can be taken:

$$I_1^s = \Lambda^s : \delta, \quad I_2^s = (\Lambda^s)^2 : \delta,$$
$$I_3^s = (\Lambda^s)^3 : \delta, \quad I_2^a = 2|\Lambda^a|^2,$$
$$J_1^s = M^s : \delta, \quad J_2^s = (M^s)^2 : \delta,$$
$$J_3^s = (M^s)^3 : \delta, \quad J_2^a = 2|M^a|^2.$$

The simultaneous invariants cannot participate as arguments, since Λ is related to the polar tensors and M is related to the axial tensors. At such selection, the defining equations take the form

$$\sigma \cdot R = a_1\delta + 2a_2\Lambda^s + 3a_3(\Lambda^s)^2 + 2\alpha\Lambda^a,$$
$$m = b_1\delta + 2b_2 M^s + 3b_3(M^s)^2 + 2\varepsilon M^a, \qquad (13)$$

where $a_k = \partial\Phi/\partial I_k^s$, $b_k = \partial\Phi/\partial J_k^s$ $(k = 1, 2, 3)$, and $\alpha = \partial\Phi/\partial I_2^a$ and $\varepsilon = \partial\Phi/\partial J_2^a$ are the state functions. In the theory of small strains of the Cosserat continuum, the elastic potential is determined as

$$\Phi = \frac{\lambda(I_1^s)^2}{2} + \mu I_2^s + \alpha I_2^a$$
$$- (3\lambda + 2\mu)I_1^s + \frac{\beta(J_1^s)^2}{2} + \gamma J_2^s + \varepsilon J_2^a.$$

Here, $a_1 = \lambda(I_1^s - 3) - 2\mu$, $a_2 = \mu$, $a_3 = 0$, $b_1 = \beta J_1^s$, $b_2 = \gamma$, and $b_3 = 0$.

As an example, consider the process of planar shear of an isotropic Cosserat elastic medium in the absence of body forces and moments described by the equations

$$x_1 = \xi_1, \quad x_2 = \chi\xi_1 + \xi_2, \quad x_3 = \xi_3. \qquad (14)$$

If shear velocity $\dot{\chi}$ is constant, the homogeneous stressed-strained state occurs in the medium, since forces of inertia of translational motion are absent. The matrices of the strain gradient tensor, rotation tensors, and angular velocity in the coordinate system under consideration are

$$(\nabla_\xi x)^* = \begin{pmatrix} 1 & 0 & 0 \\ \chi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\omega = \dot{\varphi}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The curvature and couple stress tensors are identically zero. In the absence of relative rotation of particles, when angle φ coincides with the angle of rotation of translational motion φ_e, the matrix

$$\Lambda = \nabla_\xi x \cdot R = \begin{pmatrix} \chi\sin\varphi + \cos\varphi & \chi\cos\varphi - \sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is symmetric; consequently, we have $\tan\varphi_e = \chi/2$. In this case, according to (13), the matrix

$$\sigma^* \cdot \nabla_\xi x = (R(a_1\delta + 2a_2\Lambda^s + 3a_3(\Lambda^s)^2 - 2\alpha\Lambda^a)\cdot\Lambda\cdot R^*) \quad (15)$$

is also symmetric; therefore, the equation of rotational motion, which involves only its antisymmetric part, takes the form $j\ddot{\varphi}_e = 0$. It is obviously valid only if the medium does not possesses the rotational inertia. In the inertial media, the condition $\varphi = \varphi_e + \varphi_r$ is satisfied, where φ_r is the angle of relative rotation considered to be small as compared to φ_e. Under this condition, accurate to the second-order terms, we have

$$\sin\varphi \approx \sin\varphi_e + \varphi_r\cos\varphi_e,$$

$$\cos\varphi \approx \cos\varphi_e - \varphi_r\sin\varphi_e,$$

$$\sqrt{\chi^2+4}R \approx \begin{pmatrix} 2 & -\chi & 0 \\ \chi & 2 & 0 \\ 0 & 0 & \sqrt{\chi^2+4} \end{pmatrix} - \varphi_r\begin{pmatrix} \chi & 2 & 0 \\ -2 & \chi & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\sqrt{\chi^2+4}\Lambda \approx \begin{pmatrix} \chi^2+2 & \chi & 0 \\ \chi & 2 & 0 \\ 0 & 0 & \sqrt{\chi^2+4} \end{pmatrix} - \varphi_r\begin{pmatrix} -\chi & \chi^2+2 & 0 \\ -2 & \chi & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Substitution of these approximations to formula (15) with the subsequent calculation of the antisymmetric part of the matrix $\sigma^* \cdot \nabla_\xi x$ yields the equation of rotational motion

$$j\ddot{\varphi}_r = -a_0\varphi_r + \frac{4j\chi\dot{\chi}^2}{(\chi^2+4)^2}, \quad (16)$$

where $a_0 = (\alpha - a_2)(\chi^2 + 4) - (a_1 + 3a_3(\chi^2 + 2)/2)\sqrt{\chi^2+4}$. The second term in the right part of Eq. (16) associated with the shear velocity describes continuous rotation of particles. This term is exactly equal to the moment of forces of inertia of translational motion. The first term is caused by elastic compliance of a medium with respect to rotational motion of particles. At the positive value of parameter a_0 dependent of shear angle, an oscillation mode of rotation is implemented in the medium. At the negative value, the oscillations vanish.

Consider the case of the quadratic stress potential of the Cosserat continuum. As upon shear we have $I_1^s = 1 + \sqrt{\chi^2+4}$, for this potential we have

$$a_1 = \lambda\sqrt{\chi^2+4} - 2(\lambda+\mu), \quad a_2 = \mu, \quad a_3 = 0,$$

$$a_0 = (\alpha - \lambda - \mu)(\chi^2+4) + 2(\lambda+\mu)\sqrt{\chi^2+4}.$$

At small χ, parameter a_0 is $4\alpha > 0$. The general solution of Eq. (16)

$$\varphi_r(t) = C_1\sin\frac{2\pi t}{T} + C_2\cos\frac{2\pi t}{T}$$

describes in this case the periodic natural oscillations with period T.

For most materials with the moment properties reported in literature, the quantity $\lambda + \mu$ is higher than α by an order of magnitude. Therefore, parameter a_0 positive at sufficiently small shears changes its sign. This occurs at the critical χ value

$$\chi_* = \frac{2\sqrt{2\alpha(\lambda+\mu) - \alpha^2}}{\lambda+\mu-\alpha}.$$

Upon approaching this value, the oscillatory mode of rotation of particles changes to smooth oscillationless motion. For example, the respective critical values for foamy and synthetic polyurethane are $\chi_* = 0.26$ and $\chi_* = 0.55$.

The obtained solution illustrates the main qualitative feature of stain of a medium with the microstructure as compared to an ordinary elastic medium. At the certain conditions, the shear process in such a medium is accompanied by normal oscillations of rotational motion of particles, which leads to the occurrence of acoustic resonance under certain perturbation conditions.

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