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ACOUSTIC SIGNAL PROCESSING AND COMPUTER SIMULATION

Velocities, Dispersion, and Energy of SH-Waves in Anisotropic Laminated Plates[1]

Irini Djeran-Maigre[a] and S. V. Kuznetsov[b]

[a]*INSA de Lyon, RGCU, Domaine Scientifique de la Doua, Bat. Coulomb, 20 Avenue Albert Einstein, Villeurbanne Cedex, 69621 France*

[b]*Institute for Problems in Mechanics, pr. Vernadskogo 101, Moscow, 119526 Russia*

e-mail: irini.djeran-maigre@insa-lyon.fr; kuzn-sergey@yandex.ru

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Abstract—A mathematical model is worked out for analyzing propagation ability and the specific energy of horizontally polarized shear surface waves (SH-waves) in multilayered plates. All the layers are assumed to have monoclinic symmetry. Different types of boundary conditions imposed on the outer surfaces of plates are considered. Analytic solutions for one- and two-layered plates are presented.

Keywords: SH-wave, shear wave, surface wave, Love wave, anisotropy, laminated plate, dispersion, specific energy

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1. INTRODUCTION

Horizontally polarized shear surface waves (SH-waves) propagating in multilayered plates resemble Love waves [1] in polarization, but differ in absence of a contacting half-space (substrate), and, hence excluding necessity to impose Sommerfeld's emission condition:

$$\mathbf{u}(\mathbf{x}, t) = O(|x'|^{-1}), \quad |x'| \to \infty, \tag{1}$$

where $\mathbf{u}$ is the displacement field in the substrate; $x' \equiv \boldsymbol{\nu} \cdot \mathbf{x}$ is the coordinate along depth of the substrate, and $\boldsymbol{\nu}$ is the unit normal to the plane boundary of the substrate.

As will be shown later, absence of condition (1) results in a different behavior of the SH-waves in layered plates comparing to Love waves. For example, there is the existence inequality [2] for a genuine Love wave propagating in an isotropic traction-free layer contacting with isotropic substrate:

$$(c^T_{\mathbf{nm}})_{layer} < (c^T_{\mathbf{nm}})_{substrate}, \tag{2}$$

where $c^T_{\mathbf{nm}}$ denotes speed of the corresponding shear bulk wave propagating in $\mathbf{n}$, and polarized in $\mathbf{m}$ direction. Violating inequality (2) prevents Love wave to exist. At the same time, SH-waves in two-layered plates exist at any admissible physical and geometrical properties of *isotropic* layers and at traction-free, clamped, or mixed boundary conditions (one outer surface is traction free, and the other one is clamped).

Concerning energy of the surface acoustic waves, the first works [3–7] on derivation of expressions for kinetic and elastic specific energy revealed that in contrast to bulk waves, for which kinetic and elastic specific energy coincide, in the case of surface acoustic waves these energies differ. Other theoretical studies of the surface wave energy in elastic and piezoelectric media are contained in recent works [8–11]. The analysis presented below shows that the difference between these specific energies is associated with the non-uniform distribution of the magnitude of a surface wave.

The main method used for constructing both analytic and numerical solutions for the considered SH-waves, is based on a combination of a complex formalism [12] and the modified transfer matrix (MTM) method [13, 14]. The latter being rather fast and numerically stable allows us to construct analytical solutions for plates containing several layers. These methods are also applied to analyzing specific energy of SH-waves.

2. BASIC NOTATIONS

All the regarded layers of a plate are assumed homogeneous, anisotropic and linearly hyperelastic. Equations of motion for homogeneous anisotropic elastic medium can be written in the form:

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \mathrm{div}_x \mathbf{C} \cdot \cdot \nabla_x \mathbf{u} - \rho \ddot{\mathbf{u}} = 0, \tag{3}$$

where ρ is the material density, and $\mathbf{C}$ is the elasticity tensor assumed to be *positive definite*:

$$\underset{\mathbf{A} \in sym(R^3 \otimes R^3),\, \mathbf{A} \neq 0}{\forall \mathbf{A}} \qquad (\mathbf{A} \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{A}) \equiv \sum_{i,j,m,n} A_{ij} C^{ijmn} A_{mn} > 0. \tag{4}$$

[1] The article is published in the original.

Remark 2.1. (a)—The other assumption concerns symmetry of the elasticity tensor. It will be assumed that all the regarded materials possess planes of elastic symmetry coinciding with the sagittal plane $\mathbf{m} \cdot \mathbf{x} = 0$, where vector $\mathbf{m}$ is the polarization vector of the SH-wave. This is achieved by the elasticity tensor belonging to the *monoclinic* system, and the latter is equivalent to vanishing all of the decomposable components of the tensor $\mathbf{C}$ having odd number of entries of the vector $\mathbf{m}$ in the orthogonal basis in R^3 generated by the vector $\mathbf{m}$ and any two orthogonal vectors belonging to the sagittal plane.

(b)—It will be shown later that assuming monoclinic symmetry provides a sufficient condition for the surface tractions acting on any plane $\mathbf{v} \cdot \mathbf{x} = \text{const}$ to be collinear with vector $\mathbf{m}$.

Following [15, 16], we will seek a horizontally polarized shear wave in a layer in the form:

$$\mathbf{u}(\mathbf{x}) = \mathbf{m} f(irx')e^{ir(\mathbf{n} \cdot \mathbf{x} - ct)}, \tag{5}$$

where coordinate $x' = \mathbf{v} \cdot \mathbf{x}$ is as defined in; f is the unknown scalar complex-valued function; the exponential multiplier $ir(\mathbf{n} \cdot \mathbf{v} - ct)$ in (5) corresponds to propagation of the plane wave front along direction $\mathbf{n}$ with the phase speed c; r is the wave number.

Remark 2.2. The displacement field defined by (5) is generally complex. In reality, either real or imaginary part of the right-hand side of (5) represents physical displacement field that will be implicitly assumed in the subsequent analysis. However, retaining complex expressions for the displacement field, will allow us to describe situations with the phase shift in a more convenient manner.

Substituting representation (5) into Eq. (3) and taking into account Remark 2.1. (a) yields the following differential equation:

$$\begin{aligned} &((\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \mathbf{m}) f''_{x'} \\ &+ 2(\mathbf{m} \cdot sym(\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}) \cdot \mathbf{m}) f'_{x'} \\ &+ (\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m} - \rho c^2) f) = 0. \end{aligned} \tag{6}$$

Characteristic equation for the differential equation (6), known also as the Christoffel equation, has the form:

$$\begin{aligned} &(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma^2 + 2(\mathbf{m} \cdot sym(\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}) \cdot \mathbf{m})\gamma \\ &+ (\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m} - \rho c^2) = 0. \end{aligned} \tag{7}$$

Left-hand side of Eq. (7) represents a polynomial of degree 2 with respect to the Christoffel parameter γ. Thus, for the monoclinic elastic symmetry only two partial waves form the regarded SH-wave in a layer.

The following lemma flows out from solving the Cauchy problem for Eq. (6):

Lemma 2.1. A necessary and sufficient condition for the real-analytic solution of Eq. (6), to be a nonzero function, is a simultaneous non-vanishing f and its first derivative at some x'.

Remark 2.3. For an orthotropic medium and the SH-wave propagation in a direction satisfying the principle elasticity, Eq. (7) is simplified one:

$$\begin{aligned} &(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma^2 \\ &+ (\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m} - \rho c^2) = 0. \end{aligned} \tag{8}$$

Its solution is follows:

$$\gamma_{1,2} = \pm\sqrt{\frac{\rho c^2 - \mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m}}{\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \mathbf{m}}}. \tag{9}$$

For the considered case, the general solution of Eq. (6) can be represented in the form:

$$f(irx') = C_1 \sin(r\gamma x') + C_2 \cos(r\gamma x'), \tag{10}$$

where γ is generally complex root with positive sign in (9).

3. ENERGY OF SH-WAVES

3.1. Specific Kinetic and Elastic (Potential) Energy

Herein, we derive expressions for specific kinetic and elastic (potential) energy of the SH-waves. Taking into account representation (5) and assuming $|\mathbf{m}| = 1$, the specific kinetic energy can be defined by:

$$E_{kin} \equiv \frac{1}{2}\rho \dot{\mathbf{u}} \cdot \bar{\dot{\mathbf{u}}} = \frac{1}{2}\rho \omega^2 |\mathbf{m}|^2 |f|^2, \tag{11}$$

where the following relation between the phase speed and frequency is used:

$$\omega = rc. \tag{12}$$

Equations (6), (11), and (12) allow us to represent the specific kinetic energy in the form:

$$\begin{aligned} E_{kin} \equiv &\frac{1}{2} r^2 \bar{f} \\ &\times [(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{v} \otimes \bar{\mathbf{m}}) f'' \\ &+ 2(\mathbf{m} \cdot sym(\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}) \cdot \mathbf{m}) f' \\ &+ (\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{n} \otimes \mathbf{m}) f]. \end{aligned} \tag{13}$$

Another useful expression flows out from (11) and (12):

$$\omega^2 = \frac{2E_{kin}}{\rho |f|^2}. \tag{14}$$

Now, the specific elastic energy can be defined by:

$$\begin{aligned} E_{elast} &\equiv \frac{1}{2}\nabla \mathbf{u} \cdot\cdot \mathbf{C} \cdot\cdot \nabla \bar{\mathbf{u}} \\ &= \frac{1}{2} r^2 \mathbf{m} \otimes (f'\mathbf{v} + f\mathbf{n}) \cdot\cdot \mathbf{C} \cdot\cdot \overline{(f\mathbf{n} + f'\mathbf{v})} \otimes \mathbf{m}. \end{aligned} \tag{15}$$

Remark 3.1. (a)—In view of Remark 2.2, expressions (11) and (15) coincide with the corresponding expressions for kinetic and elastic specific energy, obtained without using complex displacement fields.

(b)—Analysis of expressions (2.1), (2.3), (2.6), and (2.8) reveals that for the regarded waves $E_{kin} \neq E_{elast}$, due to presence of the generally non-constant function f. At the same time, for bulk waves $f = \text{const}$, and hence from (3) and (5) we arrive at $E_{kin} = E_{elast}$; see also [2, 7] for discussions.

Proposition 3.1. *(a)—If at some finite value of the phase speed the corresponding frequency ω vanishes, then both specific kinetic and elastic energies vanish also.*

(b)—The specific kinetic energy vanishes at any $x' \equiv \mathbf{v} \cdot \boldsymbol{x} = const$, on a plane ω, if function f vanishes at x'.

(c)—The specific elastic energy does not vanish at any finite value of the phase speed and any non-vanishing frequency ω.

Proofs of conditions (a) and (b) are obvious. Proof of condition (c) follows from the positive definite condition for the elasticity tensor, Lemma 2.1, and expressions (2.8), (12), and (15).

3.2. Group Speed

Herein, the vector-valued group speed $\mathbf{v}_{group}$ is defined by [2]:

$$\mathbf{v}_{group} = \nabla_{(r\mathbf{n})}\omega, \tag{16}$$

where $\nabla_{(r\mathbf{n})}$ denotes gradient with respect to the independent spatial variable $(r\mathbf{n})$. For the subsequent analysis the scalar group speed c_{group} will also be needed:

$$c_{group} \equiv |\mathbf{v}_{group}| = \sqrt{\nabla_{(r\mathbf{n})}\omega\overline{\nabla_{(r\mathbf{n})}\omega}}. \tag{17}$$

Now, combining (13), (14), and (17) yields:

$$c_{group} = \frac{\sqrt{(f'\mathbf{v} + f\mathbf{n})(\mathbf{m}\cdot\mathrm{C}\cdot\mathbf{m})^2\cdot(\overline{f\mathbf{n} + f'\mathbf{v}})}}{c\rho|f|}. \tag{18}$$

Where, as before, c stands for the phase speed.

Proposition 3.2. *(a)—At any physically admissible properties of a medium and any SH-wave propagating with the finite phase speed $c \neq 0$, the corresponding group speed c_{group} is delimited from zero.*

(b)—If $f \to 0$ at $x' \to x_0'$, where x_0' takes some finite value, then $c_{group} \to 0$.

Proof (a) flows out from observation that the radicand in (18) is strictly positive due to (4) and Lemma 2.1. Proof (b) is obvious.

Remark 3.2. According to Definition (17) the group speed c_{group} cannot be negative, since according to (17) c_{group} is defined as the length of the (possibly complex) vector. However, there are other definitions for the group speed, that allows negative values for c_{group}; see [17–19], where the following definition is adopted:

$$c_{group} = \frac{\partial\omega}{\partial r}. \tag{19}$$

A more detailed analysis [7] of expressions (16)–(19) reveals that the latter expression yields projection of the vector valued velocity (16) onto the wave normal $\mathbf{n}$. This provides the explanation of the possible appearing negative values of the group speed.

3.3. Ray Speed

The vector-valued ray speed can be defined by (see [7]):

$$\mathbf{v}_{ray} = \frac{\mathbf{J}_{elast}}{E_{kin} + E_{elast}}, \tag{20}$$

where $\mathbf{J}_{elast}$ is the flux of elastic energy:

$$\mathbf{J}_{elast} \equiv \dot{\mathbf{u}}\cdot\mathbf{C}\cdot\cdot\nabla\mathbf{u}. \tag{21}$$

The corresponding scalar ray speed is:

$$c_{ray} \equiv |\mathbf{v}_{ray}| = \frac{\sqrt{\mathbf{J}_{elast}\cdot\overline{\mathbf{J}_{elast}}}}{E_{kin} + E_{elast}}. \tag{22}$$

Substituting (5) into (22) and exploiting (13), (15), yields:

$$c_{ray} = \frac{2c|f|\sqrt{(f'\mathbf{v} + f\mathbf{n})\cdot(\mathbf{m}\cdot\mathbf{C}\cdot\mathbf{m})^2\cdot(\overline{f\mathbf{n} + f'\mathbf{v}})}}{\rho c^2|f|^2 + (f'\mathbf{v} + f\mathbf{n})\cdot(\mathbf{m}\cdot\mathbf{C}\cdot\mathbf{m})\cdot(\overline{f\mathbf{n} + f'\mathbf{v}})}. \tag{23}$$

Proposition 3.3. *(a)—At any physically admissible properties of a medium and any SH-wave propagating with the finite phase speed $c \neq 0$, the corresponding ray speed c_{ray} is delimited from zero.*

(b)—If $f \to 0$ at $x' \to x_0'$, where x_0' is finite, then $c_{ray} \to \infty$.

(c)—A necessary and sufficient condition for $c_{group} = c_{ray}$, is as follows:

$$(E_{kin} + E_{elast}). \tag{24}$$

Proofs (a) and (b) are analogous to the proof of Pro-position 3.2. Proof (c) follows directly from (18), (23), with account of (11), (15).

4. SINGLE–LAYERED ORTHOTROPIC PLATE

Hence it will be assumed that vectors $\mathbf{v}$, $\mathbf{m}$, and $\mathbf{n}$ coincide with the axes of elastic symmetry of an orthotropic medium.

Remark 4.1. It can be shown (see [13, 14], where similar arguments are applied to analysis of Love waves) that regardless of boundary conditions and at imaginary roots of Eq. (8), no SH-wave can propagate in directions of elastic symmetry of an orthotropic single-layered plate. Thus, the following inequality

$$c > \sqrt{\frac{\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m}}{\rho}}, \tag{25}$$

naturally arising from (9), delivers a necessary condition for existing surface SH-wave. Thus, for the regarded plate all surface SH-waves are necessary supersonic, since the radicand in the right-hand side of (25) defines speed of the corresponding shear bulk wave $c^T_{\mathbf{nm}}$. In this section we assume that condition (25) holds.

4.1. Traction-free Plate

Herein we consider a single-layered plate with the traction-free boundary conditions:

$$\begin{cases} \mathbf{t}_\nu(h/2) = 0 \\ \mathbf{t}_\nu(-h/2) = 0 \end{cases}, \tag{26}$$

where h is the thickness of the plate (we choose origin of coordinates at the median plane).

For such a plate, finding function f from (8), (26), yields:

$$f(irx') = \begin{cases} \cos(r\gamma x'), \text{ at } r = \dfrac{2n\pi}{\gamma h} \\ \sin(r\gamma x'), \text{ at } r = \dfrac{(2n-1)\pi}{\gamma h} \end{cases} \quad n = 1, 2, \dots, \tag{27}$$

where γ is defined by (9).

Proposition 4.1. *(a)—On planes $x' = const$, where*

$$x' = \begin{cases} \dfrac{\frac{1}{2}+k}{2n}h, \text{ at } r = \dfrac{2n\pi}{\gamma h}, \quad -n \le k < n \\ \dfrac{k}{2n-1}h, \text{ at } r = \dfrac{(2n-1)\pi}{\gamma h}, \quad -n \le k < n \end{cases} \quad n, k \in Z, \tag{28}$$

the displacement field and specific kinetic energy vanish. That is equivalent to existence of the internal immovable layers under propagating SH-wave on a traction-free plate.

(b)—At any finite phase speed satisfying inequality (25), there are no waves propagating at vanishing frequency (both phase speed and frequency are delimited from zero).

Proof (a)—follows from considering zeroes of the function, defined by (27). Proof (b)—follows from analyzing expressions (27), (12). It reveals that no non-trivial solutions exist at $\omega = 0$.

4.2. Clamped Plate

For a single-layered plate with clamped outer surfaces, boundary conditions are:

$$\begin{cases} \mathbf{u}(h/2) = 0 \\ \mathbf{u}(-h/2) = 0 \end{cases}. \tag{29}$$

Finding function f from Eq. (6) and satisfying boundary conditions (29), yields:

$$f(irx') = \begin{cases} \sin(t\gamma x'), \text{ at } r = \dfrac{2n\pi}{\gamma h} \\ \cos(r\gamma x'), \text{ at } r = \dfrac{(2n-1)\pi}{\gamma h} \end{cases} \quad n = 1, 2, \dots \tag{30}$$

Similarly to Proposition 4.1, we have

Proposition 4.2. (a)—On planes x' = const, where

$$x' = \begin{cases} \dfrac{k}{2n}h, \text{ at } r = \dfrac{2n\pi}{\gamma h}, \quad -n \le k \le n \\ \dfrac{\frac{1}{2}+k}{2n-1}, \text{ at } r = \dfrac{(2n-1)}{\gamma h}, \quad -n-1 \le k \le n-1 \end{cases} \quad n, k \in Z \tag{31}$$

both the displacement field and specific kinetic energy vanish. That is equivalent to existence of the internal immovable layers under propagating surface SH-wave on a clamped plate.

(b)—At any finite phase speed satisfying inequality (25), there are no waves propagating at vanishing frequency (both phase speed and frequency are delimited from zero).

4.3. Plate with Mixed Boundary Conditions

Herein we consider a plate with traction-free upper and clamped bottom surface:

$$\begin{cases} \mathbf{t}_\nu(h/2) = 0 \\ \mathbf{u}(-h/2) = 0 \end{cases}. \tag{32}$$

Direct analysis reveals that function f satisfying homogeneous boundary conditions (32) takes the form:

$$f(irx') = \begin{cases} \sin\left(r\gamma x' - \frac{\pi}{4}\right), \text{ at } r = \dfrac{2\left(n - \frac{1}{4}\right)\pi}{\gamma h}, n = 1, 2, \ldots \\ \sin\left(r\gamma x' + \frac{\pi}{4}\right), \text{ at } r = \dfrac{2\left(n + \frac{1}{4}\right)\pi}{\gamma h}, n = 0, 1, \ldots \end{cases} \quad (33)$$

Proposition 4.3. *(a)—On planes $x' = const$,* where

$$x' = \begin{cases} \dfrac{\left(k + \frac{1}{4}\right)}{2\left(n - \frac{1}{4}\right)} h, \text{ at } r = \dfrac{2\left(n - \frac{1}{4}\right)\pi}{\gamma h}, \quad -n \le k < n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n, k \in Z, \\ \dfrac{\left(k - \frac{1}{4}\right)}{2\left(n + \frac{1}{4}\right)} h, \text{ at } r = \dfrac{2\left(n + \frac{1}{4}\right)\pi}{\gamma h}, \quad -n \le k < n \end{cases} \quad (34)$$

both the displacement field and specific kinetic energy vanish. That is equivalent to existence of immovable layers under propagating surface SH-wave on a clamped plate.

(b)—At any finite phase speed satisfying inequality (25), there are no waves propagating at vanishing frequency (both phase speed and frequency are delimited from zero).

5. TWO—LAYERED ORTHOTROPIC PLATE

It is assumed that both layers are (i) orthotropic with axes of elastic symmetry coincident with vectors $\mathbf{n}$, ν, and $\mathbf{m}$; and (ii) the corresponding shear bulk waves differ:

$$(c^T_{\mathbf{nm}})_1 \neq (c^T_{\mathbf{nm}})_2. \quad (35)$$

Remark 5.1. If inequality (35) violates, then the two-layered plate becomes a single-layered, with respect to the SH-wave propagation.

5.1. Traction-Free Plate

Boundary conditions for a traction-free plate are:

$$\begin{cases} t_\nu(h_1/2) = 0 \\ t_\nu(-h_2/2) = 0 \end{cases}, \quad (36)$$

where lower indicies are referred to the corresponding layers.

Applying the Modified Transfer Matrix (MTM) method [13], functions f_k, $k = 1, 2$ which define the displacement field in the corresponding layeres, can be represented in the form:

$$f_k(irx') = \sin(r\gamma_k(x' + (-1)^k h_k/2)), \quad k = 1, 2 \quad (37)$$

at the wave number r satisfying the following equation:

$$\frac{(\mathbf{m}\otimes\mathbf{v}\cdot\cdot\mathbf{C}_1\cdot\cdot\mathbf{v}\otimes\mathbf{m})\gamma_1}{(\mathbf{m}\otimes\mathbf{v}\cdot\cdot\mathbf{C}_2\cdot\cdot\mathbf{v}\otimes\mathbf{m})\gamma_2}\sin(r\gamma_1 h_1) \times \cos(r\gamma_2 h_2) + \cos(r\gamma_1 h_1)\sin(r\gamma_2 h_2) = 0. \quad (38)$$

Proposition 5.1. (a)—Suppose that

$$\min((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2) < c < \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (39)$$

where $(c^T_{\mathbf{nm}})_k$ is the bulk wave speed in the corresponding layer, then on planes $x' = \text{const}$ where

$$x' = \frac{n\pi}{r\gamma_k} - (-1)^k \frac{h_k}{2}, \quad n \in Z, \quad (40)$$

and

$$\begin{cases} -Ent\left(\dfrac{r\gamma_1 h_1}{\pi}\right) \le n \le 0, \text{ if } (c^T_{\mathbf{nm}})_1 < (c^T_{\mathbf{nm}})_2 \\ 0 \le n \le Ent\left(\dfrac{r\gamma_2 h_2}{\pi}\right), \text{ if } (c^T_{\mathbf{nm}})_1 > (c^T_{\mathbf{nm}})_2 \end{cases} \quad (41)$$

and r satisfies Eq. (38)), both the displacement field and the specific kinetic energy vanish in a layer with the minimal bulk wave speed $c^T_{\mathbf{nm}}$.

(b)—Suppose that

$$c > \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (42)$$

(the phase speed is transonic for both layers), then on planes $x' = \text{const}$ where

$$x' = \frac{n\pi}{r\gamma_k} - (-1)^k \frac{h_k}{2}, \quad n \in Z, \quad (43)$$

and

$$\begin{cases} -Ent\left(\frac{r\gamma_1 h_1}{\pi}\right) \le n \le 0 \\ 0 \le n \le Ent\left(\frac{r\gamma_2 h_2}{\pi}\right) \end{cases}, \quad (44)$$

the displacement field and the specific kinetic energy vanish in both layers.

(c)—At the phase speed $c \to c_s - 0$, where

$$c_s = \sqrt{\frac{(\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C}_1 \cdot\cdot \mathbf{n} \otimes \mathbf{m})h_1 + (\mathbf{m} \otimes \mathbf{n} \cdot\cdot \mathbf{C}_2 \cdot\cdot \mathbf{n} \otimes \mathbf{m})h_2}{\rho_1 h_1 + \rho_2 h_2}}, \quad (45)$$

there is a lower mode SH-wave propagating with vanishing wave number $r \to 0$.

Proofs (a) and (b) flow out from expression (37) for functions f_k. Values for x' defined by (40) and (43), are zeroes of these functions.

To prove (c) we need to consider Eq. (38) at small r:

$$((\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_1 \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma_1^2 h_1 + ((\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_2 \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma_2^2 h_2)r + O(r^2) = 0. \quad (46)$$

Equating to zero the coefficient at r in the left-hand side of Eq. (46), we arrive at the solution for the phase speed given by (45).

Proposition 5.1.(c) along with expressions (11), (12), (15) ensure

Corollary. Both specific kinetic and potential energies vanish at c_s.

The typical dispersion curves (in terms of frequency and the phase speed c) for a two-layered traction-free plate are presented in figure. At the phase speed c_s both the (specific) kinetic and potential energies vanish.

Remark 5.2. (a)—Propositions 5.1.(a) and 5.1.(b) ensure that planes with vanishing kinetic energy arise only if the phase speed becomes transonic for the corresponding layer.

(b)—Direct analysis reveals that the wave speed c_s satisfies the inequalities:

$$\min((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2) < c_s < \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (47)$$

(c)—At speed c_s there can be other higher mode SH-waves propagating with non-vanishing wave numbers; see figure.

(d)—The SH-waves in the vicinity of the limiting SH-wave resemble solitones, since the coresponding wave number tends to zero as $c \to c_s - 0$.

5.2. Clamped Plate

Boundary conditions for a clamped plate are:

$$\begin{cases} \mathbf{u}(h_1/2) = 0 \\ \mathbf{u}(-h_2/2) = 0 \end{cases}. \quad (48)$$

Application of the Modified Transfer Matrix (MTM) method gives functions f_k, $k = 1, 2$ in the form:

$$f_k(irx') = \sin(r\gamma_k(x' + (-1)^k h_k/2)), \quad k = 1, 2. \quad (49)$$

The wave number in representation (49) satisfies the following equation:

$$\frac{(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_2 \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma_1}{(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_1 \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma_1}\sin(r\gamma_1 h_1) \times \cos(r\gamma_2 h_2) + \cos(r\gamma_1 h_1)\sin(r\gamma_2 h_2) = 0. \quad (50)$$

Similarly to the preceding case, we have

Proposition 5.2. (a)—Suppose that

$$\min((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2) < c < \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (51)$$

then on planes $x' = \text{const}$ where

$$x' = \frac{n\pi}{r\gamma_k} - (-1)^k \frac{h_k}{2}, \quad n \in Z, \quad (52)$$

and

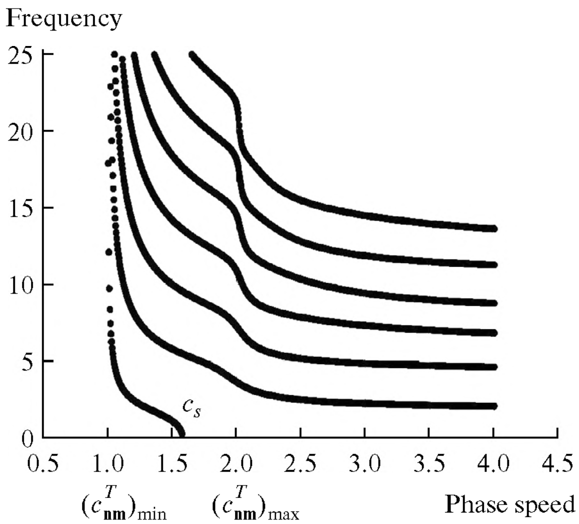
$$\begin{cases} -Ent\left(\frac{r\gamma_1 h_1}{\pi}\right) \le n \le 0, \text{ if } (c^T_{\mathbf{nm}})_1 < (c^T_{\mathbf{nm}})_2 \\ 0 \le n \le Ent\left(\frac{r\gamma_2 h_2}{\pi}\right), \text{ if } (c^T_{\mathbf{nm}})_1 > (c^T_{\mathbf{nm}})_2 \end{cases} \quad (53)$$

and r satisfies Eq. (38), both the displacement field and the specific kinetic energy vanish in a layer with the minimal bulk wave speed $c^T_{\mathbf{nm}}$.

(b)—Suppose that

$$c > \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (54)$$

(the phase speed is transonic for both layers), then on planes $x' = \text{const}$ where



Typical dispersion curves for a two-layered traction-free plate with $(c^T_{\mathbf{nm}})_{\min} = 1$, $c_s = 1.59$, and $(c^T_{\mathbf{nm}})_{\max} = 2$.

$$x' = \frac{n\pi}{r\gamma_k} - (-1)^k\frac{h_k}{2}, \quad n \in Z \quad \text{and}$$

$$\begin{cases} -Ent\left(\dfrac{r\gamma_1 h_1}{\pi}\right) \le n \le 0, \text{ at } k = 1 \\ 0 \le n \le Ent\left(\dfrac{r\gamma_2 h_2}{\pi}\right), \text{ at } k = 2 \end{cases}, \quad (55)$$

the displacement field and specific kinetic energy vanish in both layers.

(c)—At the phase speed $c \to c_s - 0$, where c_s satisfies Eq. (45), there is a lower mode SH-wave propagating with vanishing wave number $r \to 0$.

5.3. Plate with Mixed Boundary Conditions

Boundary conditions for the considered plate are:

$$\begin{cases} t_{\mathbf{v}}(h_1/2) = 0 \\ \mathbf{u}(-h_2/2) = 0 \end{cases}. \quad (56)$$

Application of the Modified Transfer Matrix (MTM) method gives functions f_k, $k = 1, 2$ in the form:

$$f_k(irx') = \sin(r\gamma_k(x' + (-1)^k h_k/2)), \quad k = 1, 2. \quad (57)$$

For the considered case, the wave number satisfies the following equation:

$$\frac{(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_2 \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma_1}{(\mathbf{m} \otimes \mathbf{v} \cdot\cdot \mathbf{C}_1 \cdot\cdot \mathbf{v} \otimes \mathbf{m})\gamma_1}\cos(r\gamma_1 h_1) \times (\cos(r\gamma_2 h_2) - \sin(r\gamma_1 h_1)\sin(r\gamma_2 h_2)) = 0. \quad (58)$$

Similarly to the preceding cases, we have

Proposition 5.3. (a)—Suppose that

$$\min((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2) < c < \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (59)$$

then on planes $x' = \text{const}$ where x' satisfies Eqs. (52), (53), both the displacement field and specific kinetic energy vanish in a layer with the minimal bulk wave speed $c^T_{\mathbf{nm}}$.

(b)—Suppose that

$$c > \max((c^T_{\mathbf{nm}})_1; (c^T_{\mathbf{nm}})_2), \quad (60)$$

(the phase speed is transonic for both layers), then on planes $x' = \text{const}$ where x' satisfies Eqs. (55), the displacement field and specific kinetic energy vanish in both layers.

(c)—At the phase speed $c \to c_s - 0$, where c_s satisfies Eq. (45), there is a lower mode SH-wave propagating with vanishing wave number $r \to 0$.

6. CONCLUDING REMARKS

Considering specific energy, it was proved that kinetic and elastic energy of SH waves generally differ; they coincide if only if the displacement distribution is uniform at the cross section of a plate.

For SH waves the explicit expressions for the group and ray speeds were derived; it was shown that both group and ray speeds defined by Eqs. (17) and (22) are positive and delimited from zero.

For monoclinic and homogeneous plates and all the considered boundary conditions: (i) the admissible speed interval is transonic:

$$c \in (c^T_{\mathbf{nm}}; \infty); \quad (61)$$

(ii) at any phase speed satisfying (61) there are immovable longitudinal layers, and (iii) there are no limiting SH-waves corresponding to the vanishing frequency.

For the two-layered monoclinic plates and all the considered boundary conditions: (i) the admissible speed interval is partly transonic

$$c \in (\min((c^T_{\mathbf{nm}})_1, (c^T_{\mathbf{nm}})_2);\infty); \quad (62)$$

(ii) at any phase speed satisfying (62) there are immovable longitudinal layers, and (iii) there are limiting SH-waves at $c \to c_s - 0$.

Existence of the limiting SH-waves at $c \to c_s - 0$ for a two-layered plate resembles Love waves propagating with vanishing frequency at speed $c \to (c^T_{\mathbf{nm}})_{substrate}$; see [13]. But, in contrast to the limiting SH-waves, Love waves at $c \to (c^T_{\mathbf{nm}})_{substrate}$; are of limited interest as they becoming leakage in the substrate [13].

It should also be noted that energy considerations associated with propagation of the surface acoustic waves not restricted to SH and Love waves, were analyzed in [20–22], see also some other works related to the discussed material [23–25].

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