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CLASSICAL PROBLEMS
OF LINEAR ACOUSTICS AND WAVE THEORY

Long Waves Induced Motions to Rigid Spheroids[1]

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Abstract—Responses of unconstrained and rigid spheroidal bodies subjected to long sound waves are analyzed by means of approaching hydrodynamic method. It is shown that in the low-frequency approximation the amplitude of translational velocity is completely determined by the density as well as the acoustic added mass which is equal to hydrodynamic one associated with the body. The inconformity of responses to sound waves in virtue of geometric asymmetry is also presented. In addition, rotational movement engendered by acoustic oblique incidence is discussed, and it represents as the modulated angular oscillation similar to the beat-frequency vibration. All these analyses on acoustically induced motions provide a theoretical evidence for developing spheroidal inertial vector receivers.

Keywords: acoustically induced motion, rigid spheroids, added mass, vector receivers

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INTRODUCTION

The acoustically forced motions to rigid bodies by low-frequency sound waves in an inviscid unbounded fluid medium have been serving as the theoretical basis of inertial acoustic vector receivers [1, 2]. Absolutely, the responses to acoustic waves of common geometric bodies, such as spheres and cylinders, have been well known when subjected to a sound plane wave. So a neutrally buoyant sphere will respond with an amplitude that is the same as for the acoustic wave [3–5]. However, as for rigid spheroidal bodies, the researches on the acoustical responses are considerably limited and more compendious [6], while the fluid flow driven motions have been perfectly investigated in the domain of hydrodynamics [7, 8]. Some preliminary works have been completed by us through numerical simulation [9], and this paper, on the basis of sound radiation from spheroids [10, 11], is devoted to providing a theoretical analysis of prolate and oblate spheroids motions due to the low-frequency underwater acoustic field, which is extremely significant for designing acoustic vector receivers of spheroidal shape.

VIBRATION AND ACOUSTIC RADIATION

Consider a rigid body of surface S in an unbounded ideal fluid medium that translates at velocity $\mathbf{U}(t)$ and rotates at angular velocity $\mathbf{\Omega}(t)$. Let the origin of the coordinate system coincident with the center of volume, and therefore the velocity potential on S is

$$-\frac{\partial\phi}{\partial y_n} = (\mathbf{U} + \mathbf{\Omega}\times\mathbf{r})\mathbf{n} \text{ on } S, \quad (1)$$

where $\mathbf{r}$ is the position vector relative to the center of volume, and $\mathbf{n}$ is the unit normal vector on S directed into the fluid. Accordingly the force and moment exerted on the solid by the radiated sound wave in the fluid are equal to the integrals

$$\mathbf{F} = \rho_0\frac{d}{dt}\oiint\limits_S \phi\mathbf{n}\,dS, \quad \tau = \rho_0\frac{d}{dt}\oiint\limits_S \phi(\mathbf{r}\times\mathbf{n})\,dS. \quad (2)$$

Here define ϕ_i^* to be the velocity potential produced by the fluid motion when the body translates at unit speed in the i direction without rotation, and let χ_i^* be the velocity potential produced when the body rotates at unit angular velocity about an axis in the i direction through the center of volume without translation, and then

$$-\frac{\partial\phi^*}{\partial y_n} = \mathbf{n}, \quad -\frac{\partial\chi^*}{\partial y_n} = (\mathbf{r}\times\mathbf{n}), \qquad (3)$$
$$\phi = U\phi^* + \Omega\chi^* \text{ on } S.$$

By using the added-mass tensor [8],

$$M_{ij} = -\rho_0\oiint\limits_S n_i\phi_j^*\,dS, \quad B_{ij} = -\rho_0\oiint\limits_S \chi_i^*(\mathbf{r}\times\mathbf{n})_j\,dS, \qquad (4)$$
$$C_{ij} = -\rho_0\oiint\limits_S \phi_i^*(\mathbf{r}\times\mathbf{n})_j\,dS = -\rho_0\oiint\limits_S n_i\chi_j^*\,dS,$$

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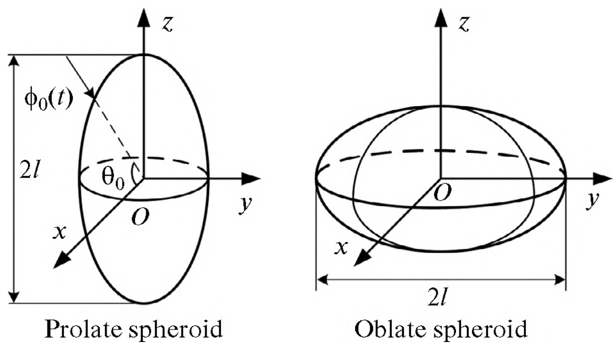

Fig. 1. Spheroids in Cartesian coordinates.

the components of force and moment of Eq. (2) can be expediently expressed as

$$F_i = -\frac{d}{dt}(U_jM_{ij} + \Omega_jC_{ij}),$$
$$\tau_i = -\frac{d}{dt}(U_jC_{ji} + \Omega_jB_{ji}), \tag{5}$$

where M_{ij} and B_{ij} denote the fluid added mass and added rotary inertia in the case of translational and rotational motion respectively, and C_{ij} represents an analogous augmentation of mass produced by rotation or an increase of rotary inertia produced by translational motion.

LONG WAVE INDUCED MOTION TO SPHEROIDS

Assume that there is an unconstrained rigid spheroid with the mass of m_s in the sound plane wave field $\phi_0(t)$, and the solid displaced fluid mass is m_0. The body will consequentially respond to the incident waves in terms of acoustic particle velocity or pressure gradient [12]. Cartesian coordinate is established to make the polar axis of spheroid coincide with z axis and equatorial axes locate on the Oxy plane, as shown in Fig. 1. Without loss of generality, incident waves are defined to radiate with an oblique angle of θ_0 and the wavefront is parallel to y axis. Since the wavelength is much longer than the overall length of the body, $kl \ll 1$ (where k is the wave number), the force (moment) from fluid is approximately equal to the integral of the pressure (cross product of pressure and the moment arm of force) on the body surface S,

$$\mathbf{f} = \oiint_S p\mathbf{n}dS, \quad \boldsymbol{\tau} = \oiint_S \mathbf{r}\times\mathbf{n}pdS,$$
$$p = \rho_0\frac{\partial\phi_0}{\partial t}. \tag{6}$$

i—Translational movement and force analysis.

The force component along x axis can be written as

$$f_x = \rho_0\frac{\partial}{\partial t}\oiint_S \phi_0\cos(n,x)dS = m_0\frac{\partial U_0}{\partial t}\cos\theta_0. \tag{7}$$

Here the velocity potential $\phi_0 = -U_0r$ represents the ideal low-frequency plane wave with the particle velocity amplitude of U_0, and $\mathbf{n}$ is the outer normal to the spheroid surface when $\cos(n, x)$ is the associated direction cosine. According to the Newton's second law of motion, there is

$$f_x + F_x = (m_s + M_{xx})\frac{\partial U_x}{\partial t},$$
$$F_x = M_{xx}\frac{\partial U_0}{\partial t}\cos\theta_0, \tag{8}$$

where U_x is the translational velocity of the spheroid along x axis, and F_x and M_{xx} are the reacting force and the accompanying added mass from fluid in the case of sound radiation of the moving body as derived from Eq. (5). Then

$$\frac{U_x}{U_0\cos\theta_0} = \frac{m_0 + M_{xx}}{m_s + M_{xx}} = \frac{1+K_x}{\rho_s/\rho_0 + K_x}, \tag{9}$$

and similarly

$$\frac{U_z}{U_0\sin\theta_0} = \frac{1+K_z}{\rho_s/\rho_0 + K_z}, \tag{10}$$

where $K_i = M_{ii}/m_0$ denotes a dimensionless coefficient of added mass. It is evident that the acoustically induced translational motion is only dominated by densities and added-mass coefficients. Apparently, motions along polar and equatorial axes present the non-negligible difference in virtue of the incongruous added-mass coefficients when the body is non-neutrally buoyant. In the low-frequency approximation, it is indicated from [5, 11] that acoustic added masses for spheroids are identical to hydrodynamic ones, which exhibit the frequency-independent closed form. For prolate spheroids, the added-mass coefficients given in [7] are expressed as

$$K_{px} = K_{py} = \frac{\alpha_p}{2-\alpha_p}, \quad K_{pz} = \frac{\gamma_p}{2-\gamma_p},$$
$$\alpha_p = \frac{1}{e^2} - \frac{1-e^2}{2e^3}\ln\frac{1+e}{1-e}, \tag{11}$$
$$\gamma_p = \frac{2(1-e^2)}{e^3}\left(\frac{1}{2}\ln\frac{1+e}{1-e} - e\right),$$

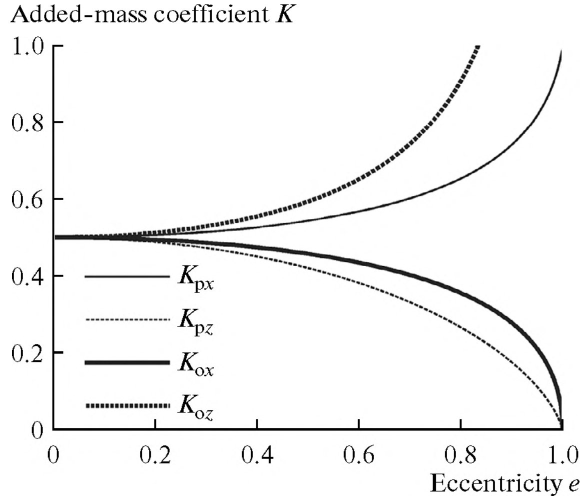

Fig. 2. Added-mass coefficients of fluid versus eccentricity in terms of translational motion of spheroids.

and for oblate spheroids they are

$$K_{ox} = K_{oy} = \frac{\alpha_o}{2-\alpha_o}(Ox), \quad K_{oz} = \frac{\gamma_o}{2-\gamma_o}(Oz),$$

$$\alpha_o = \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}, \tag{12}$$

$$\gamma_o = \frac{2}{e^2}\left[1-\sqrt{1-e^2}\frac{\sin^{-1}e}{e}\right].$$

It is obviously demonstrated from Eqs. (11), (12), and Fig. 2 that the added-mass coefficients are completely determined by the eccentricity, and when $e \longrightarrow 0$ ($K = 0.5$) the prolate spheroid and oblate spheroid convert to spheres, and when $e \longrightarrow 1$ ($K = 1$) the prolate spheroid and oblate spheroid transform into a cylinder and a thin disk respectively.

Through utilizing the exceptional characteristic of motions of spheroids that are similar to spheres, it is evidently understandable that the inertial acoustic vector receivers based on spheroids are distinctly capable of two-dimensional reception of acoustic particle velocity. However, due to the discrepancy between responses along polar and equatorial axes, the feasibility of three-dimensional reception demands further analysis.

Here, for the oblique incidence of the acoustic wave with an angle of θ_0, the arctangent arithmetic directly using velocity components is commonly employed to estimate the pitch angle when the body is used as a vector receiver, and that is

$$\tan\theta = \frac{U_z}{U_x}. \tag{13}$$

Substituting Eqs. (9) and (10) into Eq. (13), the angular deviation can be easily written as

$$\Delta\theta = \arctan\left(\frac{1+K_x}{\rho_s/\rho_0+K_x}\Big/\frac{1+K_z}{\rho_s/\rho_0+K_z}\tan\theta_0\right) - \theta_0. \tag{14}$$

Here, it is noted that the deviations emerge only for non-neutral buoyant spheroids and are also closely related to the added-mass coefficients and the degree of incident angle. Figures 3a and 3b show that when $\rho_s/\rho_0 = 1.2$ and $e \longrightarrow 1$, the maximal value of deviation is approximately 2.5° for the prolate spheroid and –5° for the oblate spheroid. For a more precise solution, the derivative operation could be performed upon Eq. (14) to obtain the maximum $\Delta\theta$ and the corresponding incident angle:

$$\theta_{0\max} = \arcsin\sqrt{\frac{1+K_z}{\rho_s/\rho_0+K_z}\Big/\left(\frac{1+K_x}{\rho_s/\rho_0+K_x}+\frac{1+K_z}{\rho_s/\rho_0+K_z}\right)}. \tag{15}$$

ii—Rotational movement and moment analysis.

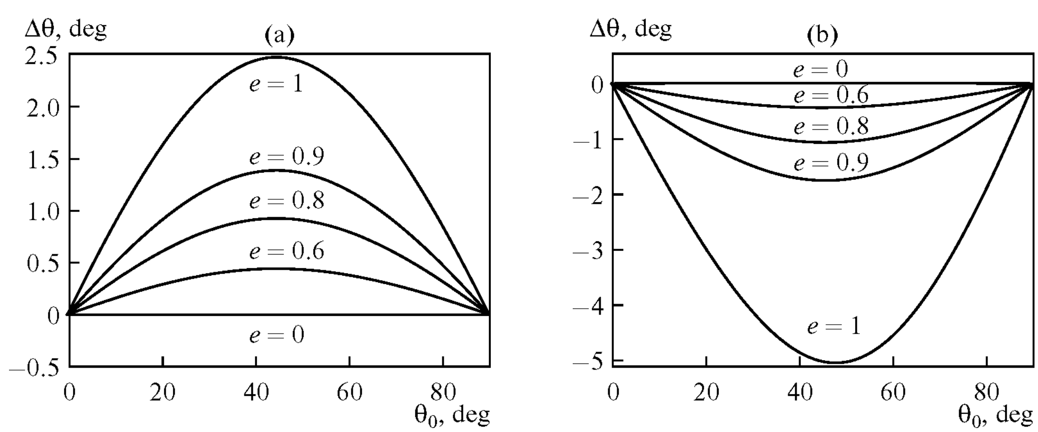

Fig. 3. (a) Estimated angle deviation for prolate spheroids and (b) oblate spheroids, when using the arctangent of velocity component U_z versus U_x in the case of obliquely incident angles from 0° to 90° with varying eccentricities, $\rho_s/\rho_0 = 1.2$.

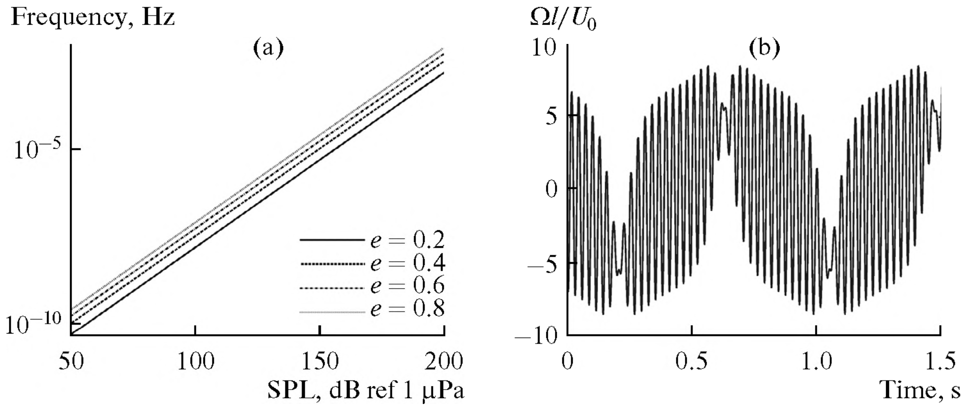

Fig. 4. (a) Relationship between the inherent frequency of rotational motion and the amplitude of sound wave, as well as eccentricity, when there is a steady velocity, and (b) is the time waveform of the angular velocity driven by harmonic waves, ($l = 0.1$ m, $e = 0.6$, $\rho_s/\rho_0 = 1$, $\theta_0 = 45°$, $U_0 = 15$ m/s, $kl = 0.015$).

According to the characteristic of inertial ellipsoid in rigid motion, the rotational oscillation could keep stable around the y axis, and therefore the moment in Eq. (6) is

$$\tau_y = \rho_0 \frac{d}{dt} \oint\!\!\oint_S (\mathbf{r} \times \mathbf{n})_y \phi_0 dS - \rho_0 \oint\!\!\oint_S (\mathbf{U}_0 \times \mathbf{n})_y \phi_0 dS, \quad (16)$$

where, as mentioned in Eqs. (2) and (5), the first integral term can be represented as

$$\rho_0 \frac{d}{dt} \oint\!\!\oint_S (\mathbf{r} \times \mathbf{n})_y \phi_0 dS = -\frac{d\Omega_y}{dt} B_{yy} \ (C_{ij} \equiv 0), \quad (17)$$

and for the second integral term, by using the added-mass tensor, the integral term can be further written as

$$\rho_0 \oint\!\!\oint_S (\mathbf{U}_0 \times \mathbf{n})_y \phi_0 dS = (M_{xx} - M_{zz}) \frac{U_{0x} U_{0z}}{2}, \quad (18)$$

where U_{0x} and U_{0z} are acoustic particle velocity components along the x axis and z axis respectively, and

$$(\mathbf{U}_0 \times \mathbf{n})_y = U_{0x} n_z - U_{0z} n_x. \quad (19)$$

For a free prolate spheroid, it evidently can execute oscillations governed by

$$(I_y + B_{yy}) \frac{d^2\theta}{dt^2} + (M_{xx} - M_{zz}) U_0^2 \frac{\sin 2\theta}{2} = 0, \quad (20)$$

where I_y is the rotary inertia of the spheroid around the y axis and B_{yy} denotes the associate added rotary inertia of the fluid. Since Eq. (20) belongs to variable-coefficient nonlinear equations, it is difficult to directly seek an analytical solution, and as a result the numerical computation was conducted based on MATLAB to obtain the approximate solution.

Firstly, for a special case of the uniform and steady velocity such as the hydrodynamic flow (U_0 is time-independent), Eq. (20) is similar to the undamped nonlinear vibration formula of a simple pendulum, and hence the accurate solution can be expressed by Legendre elliptic integral and Jacobian elliptic function.

Since the formula satisfies the condition of Hamilton system [13], we can assume that

$$\frac{d\theta}{dt} = \omega_0 \sqrt{m^2 - \sin^2\theta}, \qquad \omega_0 = U_0 \sqrt{\frac{M_{xx} - M_{zz}}{I_y + B_{yy}}}, \quad m = \sin\theta_0. \quad (21)$$

Then, the inherent frequency for the angular motion can be effortlessly obtained:

$$f = \frac{\omega_0}{2\mathrm{K}(m)}, \qquad \mathrm{K}(m) \equiv \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m^2 \sin^2\varphi}} d\varphi, \quad \sin\theta = m \sin\varphi, \quad (22)$$

where ω_0 denotes the inherent angular frequency of angular oscillation that is similar to a simple pendulum in the case of small incident angles, and $\mathrm{K}(m)$ is the Legendre complete elliptic integral of the first kind. Figure 4a demonstrates that the inherent frequency is exceptionally proportional to the amplitude of incident sound wave and the eccentricity. As for acoustic particle velocity of infinitesimal small amplitude waves, the inherent frequencies of angular oscillation tend towards zero.

In the Legendre elliptic integral of the first kind, we put $p = \sin\varphi$, and then

$$\pm\omega_0 t = \int_0^{\sin\varphi} \frac{1}{\sqrt{(1 - p^2)(1 - mp^2)}} dp, \quad (23)$$

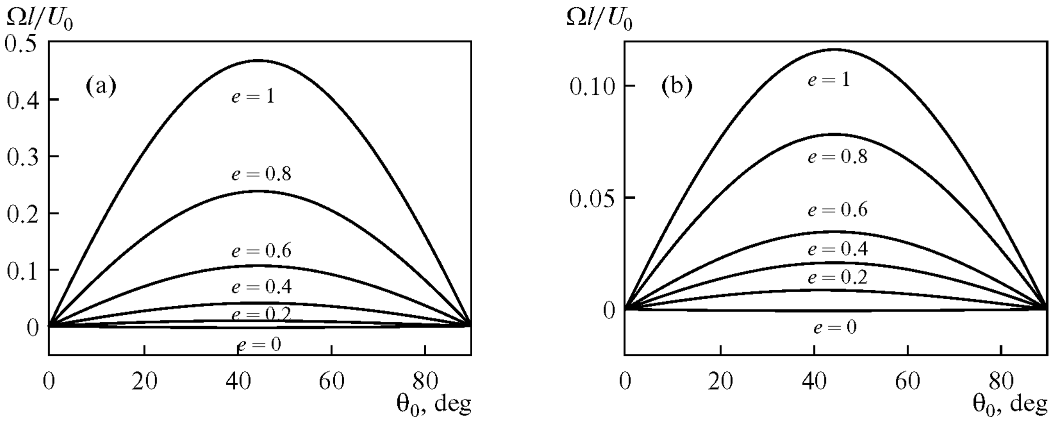

Fig. 5. (a) Initial amplitude of angular velocity of prolate spheroids and (b) oblate spheroids, when the inherent frequency is much lower compared to incident waves ($l = 0.1$ m, $\rho_s/\rho_0 = 1$, $U_0 = 1$ m/s, $kl = 0.1$).

where $\sin\varphi$ can be considered as the function of $\omega_0 t$, and hence we can get the solution through seeking the inverse function of Eq. (23):

$$\sin\varphi = \pm \mathrm{sn}(\omega_0 t, m), \tag{24}$$

where $\mathrm{sn}(\omega_0 t, m)$ indicates the Jacobian elliptic sine function. Combining Eqs. (21), (22) and (23), the angular velocity is

$$\Omega(t) = \pm 2\omega_0 m \mathrm{cn}(\omega_0 t, m) \quad (m = \sin\theta_0), \tag{25}$$

where $\mathrm{cn}(\omega_0 t, m)$ denotes the Jacobian elliptic cosine function, and $\mathrm{cn}^2 \mathrm{u} = 1 - \mathrm{sn}^2 \mathrm{u}$.

When the harmonic sound waves possess a much higher frequency than the inherent frequency of rotation, the time waveforms of motion are computed and presented in Fig. 4b, which distinctly exhibits the modulated angular oscillation resembling the beat-frequency vibration. The oscillation consists of two parts, the low-frequency swing motion with inherent frequency of rotation, and the high-frequency angular oscillation with the same frequency as the incident sound wave. Here in the computations, let $l = 0.1$ m, $e = 0.6$, $\rho_s/\rho_0 = 1$, $\theta_0 = 45°$, $U_0 = 10$ m/s, and the frequency of the sound wave satisfies $kl = 0.015$, so that the wavelengths are much longer than the overall length of the body. In order to conveniently demonstrate the level of rotational motion, the angular velocity is presented in terms of the velocity of the spheroid pole normalized on the velocity of incident sound wave, and that is $\Omega l/U_0$.

The initial amplitude of angular velocity is computed through a numerical approach, as illustrated in Fig. 5, and it can be concluded that the maximum angular velocities appear at the angle of about $\theta_0 = 45°$ and increase with the eccentricity growing up. For example, the maximal velocity ratio is approximately 0.47 for a prolate spheroid and 0.12 for an oblate spheroid when $e = 1$, $l = 0.1$ m, $\rho_s/\rho_0 = 1$, $U_0 = 1$ m/s, $kl = 0.1$.

CONCLUSIONS

Long sound waves induced motions to spheroids are analyzed through analytical derivation including translational and rotational movements. Therein, the translational amplitude responses are described by an identical compact formula that entirely consists of densities and added-mass coefficients, which consolidate the sound reception theories of spherical and cylindrical acoustic vector receivers, as well as disc-shaped ones. Spheroidal bodies respond uniformly to the horizontal two-dimensional plane sound waves, but inconsistently to vertical incidence because of the discrepancy of the added-mass coefficients, which will result in estimated angular deviations due to variant sensitivities between different channels, when there are identical vibration sensors mounted along different axes in spheroids. Consequently, accurate calibrations are definitely indispensable before acoustic vector receivers are employed as measurement tools.

Moreover, an oblique incidence could engender moments that definitely produce a nonlinear angular movement around the equatorial axes of spheroids, which presents modulated oscillation similar to the beat-frequency motion, and the inherent frequency is closely related to the strength of incident waves. Under conventional considerations, excessive care should be taken of that the rotational motion introduced centripetal acceleration is treated as disturbances that should be eliminated, and therefore the vibration sensors are commonly placed in pairs in the center of bodies with signal-difference output [12]. Otherwise, it can be also concluded that the angular velocity could bring an increment in the total velocity at the pole of spheroids, and therefore by contraries, a vector receiver could benefit from the rotational motion through embedding a pickup at the top of spheroids instead.

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