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CLASSIC PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

Numerical Implementation of Huygens Principle for Scattering from a Smooth Ideal Surface

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Abstract—In 1678 Huygens formulated a principle postulating that each point on a wave front acts as a point source emitting a spherical wave which travels with a local velocity. The field at a given point some time later is then the sum of the fields of each of these point sources [1]. In this article a numerical method is presented for 2D problems of sound propagation and scattering, conforming that physical assumptions.

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INTRODUCTION

The mathematical theory of Huygens principle, for acoustic waves was developed by Helmholttz in 1880 for harmonic waves and Kirchhoff in 1883 for transient waves [2–4]. According to that theory an exact solution for scattering harmonic acoustic waves is given by Helmholtz formula:

$$p(\mathbf{r}) = \frac{1}{4\pi}\int_S \left(p(\mathbf{q})\frac{\partial G(\mathbf{r}-\mathbf{q})}{\partial \nu} - G(\mathbf{r}-\mathbf{q})\frac{\partial p(\mathbf{q})}{\partial \nu}\right) dS,$$

where S—scattering surface, vector $\mathbf{q}$ belongs to $S, G(\mathbf{r}-\mathbf{q})$—Green function and $p(\mathbf{r})$—sound pressure. To apply this formula one must have value of sound pressure $p(\mathbf{q})$ and its normal derivative $\frac{\partial p(\mathbf{q})}{\partial \nu}$ to the surface. Usually only one of this two quantities is known on a non-flat surface. An integral equation [5] is derived for this case for unknown pressure or normal derivative. The scattering of sound from the surface is a classic problem of acoustics and in [6–9] it was investigated by perturbation technique, for the small slopes [10, 11] and estimated a validity of approaches above in [12, 13] and in parabolic approximation [14, 15]. In this article we follow [7] to fit boundary condition for each Fourier harmonic of the scattered field, omitting solution of integral equation and instead of calculation of integrals with singularities from oscillating functions we offer simple and fast method for direct calculation of the field scattered by smooth surfaces, based on parabolic equation technique.

EQUATIONS OF THE FIELD

Equations of harmonic sound field in two dimensions (x,z) are [5]:

$$-i\omega p + \rho c^2(\nabla, \mathbf{V}) = 0, \tag{1}$$

$$-i\omega\rho\mathbf{V} + \nabla p = 0, \tag{2}$$

where p—pressure, $\mathbf{V} = \mathbf{i}V_x + \mathbf{k}V_z$ particles velocity, $\omega = 2\pi F$—is a circular frequency, F—frequency, ρ—density and c—sound speed. Let us introduce a smooth enough curve, as a function of its arc length σ, located in plane $y = 0$, which will be representing either a wave front or the scattering surface:

$$\mathbf{S}(\sigma) = \mathbf{i}u(\sigma) + \mathbf{k}w(\sigma), \tag{3}$$

with tangent vector

$$\mathbf{t}(\sigma) = \mathbf{i}u_\sigma + \mathbf{k}w_\sigma, \tag{4}$$

normal vector

$$\mathbf{n}(\sigma) = -\mathbf{i}w_\sigma + \mathbf{k}u_\sigma. \tag{5}$$

Let ν be a distance along normal vector from the surface and let us build a new coordinate system (ν,σ)

$$x(\nu,\sigma) = u(\sigma) - \nu w_\sigma(\sigma), \tag{6}$$

$$z(\nu,\sigma) = w(\sigma) + \nu u_\sigma(\sigma), \tag{7}$$

$$\mathbf{V}(\nu,\sigma) = \mathbf{n}V_\nu + \mathbf{t}V_\sigma. \tag{8}$$

Lame coefficients for new coordinates read

$$h_\nu = 1, \quad h_\sigma = 1 - \kappa(\sigma)\nu, \tag{9}$$

where $\kappa(\sigma)$ is a curvature of the surface in a point σ. Differential operators from Eq. (1)–(2) take the form:

$$\nabla p = \mathbf{n}(\sigma)\frac{\partial p}{\partial \nu} + \mathbf{t}(\sigma)\frac{1}{1-\kappa(\sigma)\nu}\frac{\partial p}{\partial \sigma}, \tag{10}$$

$$(\nabla, \mathbf{V}) = \frac{\partial V_\nu}{\partial \nu} - \frac{\kappa}{h_\sigma}V_\nu + \frac{1}{h_\sigma}\frac{\partial V_\sigma}{\partial \sigma}. \tag{11}$$

Equations of motion become:

$$\frac{\partial p}{\partial \nu} = i\omega\rho V_\nu, \tag{12}$$

$$\frac{1}{h_\sigma}\frac{\partial p}{\partial \sigma} = i\omega\rho V_\sigma, \tag{13}$$

$$-i\omega\rho p + \rho c^2\left(\frac{\partial V_\nu}{\partial \nu} - \frac{\kappa}{h_\sigma}V_\nu + \frac{1}{h_\sigma}\frac{\partial V_\sigma}{\partial \sigma}\right) = 0. \tag{14}$$

Now we substitute for V_σ its expression from Eq. (13) and obtain a system

$$\mathbf{P}_\nu = \begin{pmatrix} 0 & i\omega\rho \\ -\frac{1}{i\omega\rho}\left(\frac{1}{h_\sigma}\frac{\partial}{\partial\sigma}\frac{1}{h_\sigma}\frac{\partial}{\partial\sigma} + k^2\right) & \frac{\kappa}{h_\sigma} \end{pmatrix}\mathbf{P}, \mathbf{P} = \begin{pmatrix} p \\ V_\nu \end{pmatrix}, \tag{15}$$

where $k = \frac{\omega}{c}$.

SOLUTION

Split step algorithm

Consider Eq. (15) at $\nu = 0$, or at the boubdary:

$$\mathbf{P}_\nu = \begin{pmatrix} 0 & i\omega\rho \\ -\frac{1}{i\omega\rho}\left(\frac{\partial^2}{\partial\sigma^2} + k^2\right) & \kappa(\sigma) \end{pmatrix}\mathbf{P}. \tag{16}$$

To solve system (16) we split its matrix on two matrices:

$$\hat{M} = \begin{pmatrix} 0 & i\omega\rho \\ -\frac{1}{i\omega\rho}\left(\frac{\partial^2}{\partial\sigma^2} + k^2\right) & \kappa(\sigma) \end{pmatrix} = \hat{A} + \hat{a}, \tag{17}$$

$$\hat{A} = \begin{pmatrix} 0 & i\omega\rho \\ -\frac{1}{i\omega\rho}\left(\frac{\partial^2}{\partial\sigma^2} + k^2\right) & 0 \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} 0 & 0 \\ 0 & \kappa(\sigma) \end{pmatrix}. \tag{18}$$

According to the split-step algorithm [16] to obtain solution of Eq. (16) at $\nu + d\nu$ one should perform the following operations:

$$\mathbf{P}(\sigma, \nu + d\nu) = \left(\hat{I} + \frac{d\nu}{2}\hat{A}\right)\left(\hat{I} - \frac{d\nu}{2}\hat{A}\right)^{-1} \times \left(\hat{I} + \frac{d\nu}{2}\hat{a}\right)\left(\hat{I} - \frac{d\nu}{2}\hat{a}\right)^{-1}\mathbf{P}(\sigma,\nu). \tag{19}$$

To accomplish calculations according to Eq.(19) we must find eigen-vectors of matrix $\hat{A}$.

Eigen vectors

Equations for right $\mathbf{R}$ and left $\mathbf{L}$ eigen vectors of matrix $\hat{A}$ are:

$$\begin{pmatrix} -i\xi & i\omega\rho \\ -\frac{1}{i\omega\rho}\left(\frac{\partial^2}{\partial\sigma^2} + k^2\right) & -i\xi \end{pmatrix}\mathbf{R}(\xi) = 0, \mathbf{L}(\xi)\begin{pmatrix} -i\xi & i\omega\rho \\ -\frac{1}{i\omega\rho}\left(\frac{\partial^2}{\partial\sigma^2} + k^2\right) & -i\xi \end{pmatrix} = 0.$$

Let L be a total arc length of the surface. If $0 \le \sigma \le L$, then the periodic solution with respect to σ are:

$$\mathbf{R}_m^{1,2} = \begin{pmatrix} \sqrt{\frac{\omega\rho}{2\xi_m^{1,2}}} \\ \sqrt{\frac{\xi_m^{1,2}}{2\omega\rho}} \end{pmatrix}\exp(i\mu_m\sigma), \tag{20}$$

$$\mathbf{L}_m^{1,2} = \begin{pmatrix} \sqrt{\frac{\xi_m^{1,2}}{2\omega\rho}} & \sqrt{\frac{\omega\rho}{2\xi_m^{1,2}}} \end{pmatrix}\exp(i\mu_m\sigma), \tag{21}$$

$$(\mathbf{L}_m^k, \mathbf{R}_m^l) = \delta_{kl}, \tag{22}$$

where

$$\mu_m = \frac{2m\pi}{L}, \quad \xi_m^{(1,2)} = \pm\sqrt{k^2 - \mu_m^2}, \tag{23}$$

and upper indices correspond to the sign of ξ. Only vectors with index 1 are selected with $\mathrm{Im}\xi_{1,m} \ge 0$ what eliminates the waves increasing at infinity. Terms in Eq. (15) containing matrix $\hat{a}$ can be calculated explicitly as follows:

$$\left(\hat{I} + \frac{d\nu}{2}\hat{a}\right)\left(\hat{I} - \frac{d\nu}{2}\hat{a}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1 + 0.5\kappa d\nu}{1 - 0.5\kappa d\nu} \end{pmatrix}. \tag{24}$$

Algorithm

Split-step algorithm Eq. (15) now takes the form

$$\mathbf{P}(\sigma, \nu + d\nu) = \mathbf{F}^{-1}\left(\hat{R}^1\hat{\Xi}\left(\hat{L}^1\mathbf{F}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1 + 0.5\kappa(\sigma)d\nu}{1 - 0.5\kappa(\sigma)d\nu} \end{pmatrix}\mathbf{P}(\sigma,\nu)\right)\right), \tag{25}$$

where $\hat{L}^1, \hat{R}^1$—are rectangular matrices of left and right eigen-vectors with index 1:

$$\hat{L}^1 = \begin{pmatrix} \sqrt{\frac{\xi_0^1}{2\omega\rho}} & \sqrt{\frac{\omega\rho}{2\xi_0^1}} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{\frac{\xi_1^1}{2\omega\rho}} & \sqrt{\frac{\omega\rho}{2\xi_1^1}} & .. \\ 0 & 0 & 0 & 0 & .. \\ \dots & \dots & \dots & \dots & .. \end{pmatrix},$$

$$\hat{R}^1 = \begin{pmatrix} \sqrt{\frac{\omega\rho}{2\xi_0^1}} & 0 & \dots \\ \sqrt{\frac{\xi_0^1}{2\omega\rho}} & 0 & .. \\ 0 & \sqrt{\frac{\omega\rho}{2\xi_1^1}} & .. \\ 0 & \sqrt{\frac{\xi_1^1}{2\omega\rho}} & .. \\ \dots & \dots & \dots \end{pmatrix},$$

$\mathbf{F}, \mathbf{F}^{-1}$ are direct and inverse Fourier transforms with respect to σ and

$$\hat{\Xi}_{mn} = \delta_{mn} \frac{1 + 0.5i\xi_m^1 d\nu}{1 - 0.5i\xi_m^1 d\nu}.$$

Step of integration $d\nu$ selected as $\frac{\lambda}{8}, \frac{\lambda}{16}$, where λ is a wavelength, since in most cases it limits distance where the field can be calculated. In the case of cylinder, for example, size of a coordinate cell increase linearly with distance.

Adjustment of coordinates

Let arc length along the surface at $\nu = 0$ is σ_0 and at $d\nu$ is σ_1. Then

$$h_{\sigma_0} d\sigma_0 = (1 - \kappa(\sigma_0) d\nu) d\sigma_0$$
$$= \left(1 + \frac{R_1 - R_0}{R_0}\right) d\sigma_0 = \frac{R_1}{R_0} d\sigma_0 = d\sigma_1$$

where R_0 is radius of curvature at $\nu = 0$ and R_1 at $\nu = d\nu$. That means that the form of equation (16) and solution do not change, but one must recalculate $x(\sigma_1), z(\sigma_1), \kappa(\sigma_1)$ as functions of the new arc length σ_1 from Eq. (7)–(9).

Start Values and Boundary Conditions

Let

$$\mathbf{P}_i(\sigma) = \begin{pmatrix} p_i(\sigma) \\ V_i(\sigma) \end{pmatrix}$$

from Eq. (15) represent sound pressure and normal velocity, created by incident field. Then according to approximation of boundary by tangent plane [7] we set for the reflected field from ideally soft surface

$$\mathbf{P}_r(\sigma) = \begin{pmatrix} -p_i(\sigma) \\ V_i(\sigma) \end{pmatrix}$$

so total pressure on the surface will be zero. For the case of ideally rigid boundary we set for reflected field

$$\mathbf{P}_r(\sigma) = \begin{pmatrix} p_i\sigma) \\ -V_i(\sigma) \end{pmatrix}.$$

The vectors $\mathbf{P}_r(\sigma)$ become a start values for the algorithm.

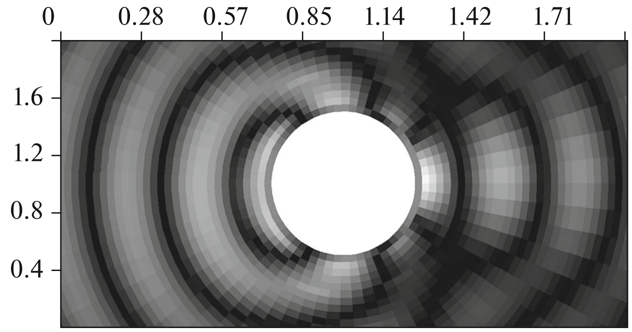

Fig. 1. Wave scattered by circle $(x = a\cos\phi, y = b\sin\phi), 0 \le \phi \le 2\pi, \ a = 1$ m, $b = 1$ m. Frequency 750 Hz, sound speed 1500 m/s, wavelength 1.5 m, density 1 g/cm^3.

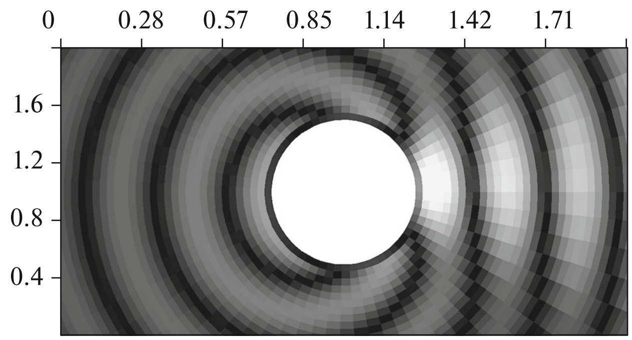

Fig. 2. Exact scattered wave by circle $(x = a\cos\phi, y = b\sin\phi), 0 \le \phi \le 2\pi, \ a = 1$ m, $b = 1$ m. Frequency 750 Hz, sound speed 1500 m/s, wavelength 1.5 m, density 1 g/cm^3.

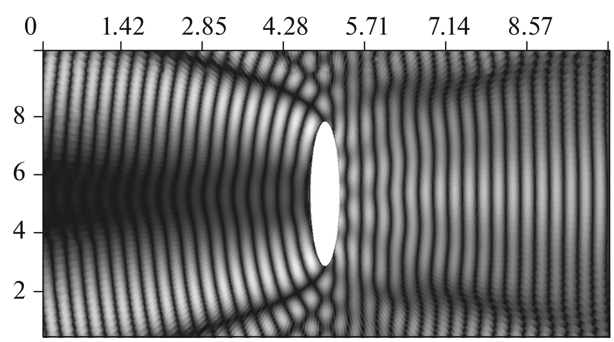

Fig. 3. Plane wave incident from the lest on acoustically soft ellipse $(x = a\cos\phi, y = b\sin\phi), 0 \le \phi \le 2\pi, a = 1$ m, $b = 5$ m. Frequency 750 Hz, sound speed 1500 m/s, wavelength 2 m, density 1 g/cm^3.

EXAMPLES

In all examples horizontal and vertical scales are expressed in terms of wavelengths. Figures 1, 2 represent comparison of solution of Eq. (17) with exact solution for scattering of plane wave on cylinder in form of series of Bessel functions from [1, page 1377]. All details of phase structure are in good correspondence. Next example at Fig. 3 illustrates scattering of plane wave on acoustically soft elliptical object and at Fig. 4 on ideally hard object. Intensity is proportional to absolute value of real part of sound pressure. Looking at scattered and total field makes clear, that

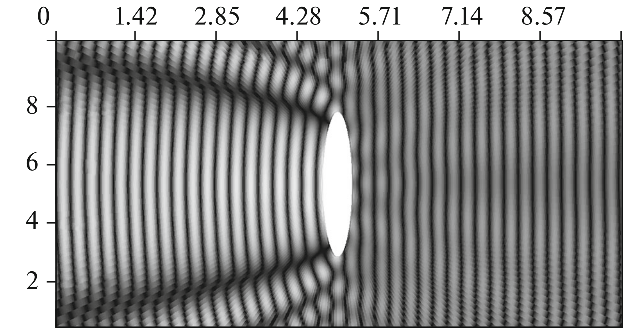

Fig. 4. Plane wave incident from the lest on acoustically hard ellipse $(x = a\cos\phi, y = b\sin\phi), 0 \le \phi \le 2\pi, a = 1$ m, $b = 5$ m. Frequency 750 Hz, sound speed 1500 m/s, wavelength 2 m, density 1 g/cm^3.

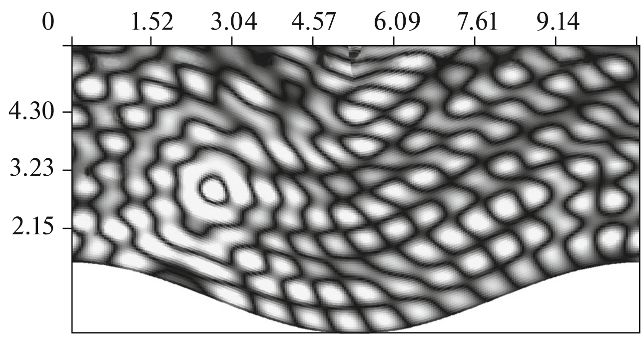

Fig. 5. Total field of point source over one 16 m periods of soft sinusoidal surface with amplitude 1 m. Frequency 1000 Hz, sound speed 1500 m/s, wavelength 1.5 m, density 1 g/cm^3.

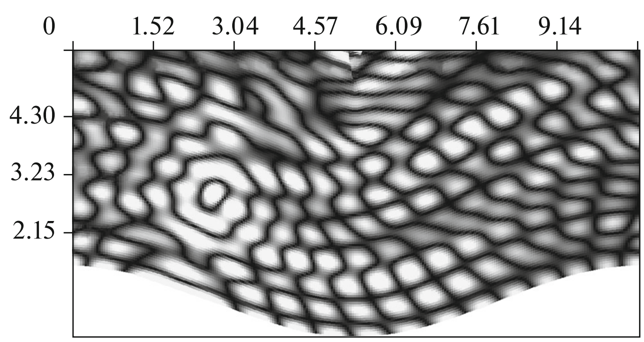

Fig. 6. Total field of point source over one 16 m periods of soft sinusoidal surface with amplitude 1 m. Frequency 1000 Hz, sound speed 1500 m/s, wavelength 1.5 m, density 1 g/cm^3.

shadow is not an absence of field, but result of interference. Last example 3 at Figs. 6, 7 illustrates scattering of an incident harmonic wave, generated by point source from acoustically soft and ideally hard sinusoidal surface at the bottom of the picture.

CONCLUSION

By reducing boundary value problem to initial value problem in curvilinear coordinates a new method is presented for computation of sound fields in two dimensions with smooth acoustically soft and acoustically hard boundaries.

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