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NONLINEAR ACOUSTICS

Oscillation and Wave Excitation in Quadratically Nonlinear Systems with Selective Second Harmonic Suppression

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Abstract—Oscillation and wave excitation processes are investigated in systems with quadratic nonlinearity in the presence of selective absorption at second harmonic frequency. Specific examples are considered: wave excitation by sources moving with velocity close to that of natural disturbances in the medium, wave excitation in a plane layer (one-dimensional resonator) by oscillation of one of the walls, and forced oscillations of two coupled oscillators. It is shown that, as the second harmonic absorption factor increases, the fundamental oscillation amplitude grows. Similar relation holds for nondispersive nonlinear waves characterized by shock wave formation and energy "spread" to higher harmonics. Second harmonic suppression "blocks" the cascade process of energy transfer toward upper part of the spectrum and "turns off" nonlinear absorption. Some of the systems under consideration had been implemented and experimentally investigated. Selectively absorbing media designed for high-frequency waves can be considered as metamaterials and synthesized on the basis of corresponding technologies.

Keywords: selective absorption, shock fronts, oscillation excitation, nonlinearity suppression

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EXAMPLES OF SYSTEMS AND SIMPLFIED EQUATIONS

Nonlinear processes studied below and equations describing them are of fairly general nature. Therefore, it is desirable to consider examples of actual problems corresponding to the adopted mathematical model. In this section, it is shown that there exists a multitude of concentrated and distributed systems that are adequately described by a pair of coupled nonlinear equations lying at the basis of subsequent analysis.

First, let us consider wave excitation by distributed sources moving with velocity close to that of natural disturbance propagation in the medium. For definiteness, consider the problem of thermooptic generation of sound with allowance for quadratic nonlinearity, which is described by inhomogeneous wave equation [1]

$$\Delta p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\varepsilon}{c_0^4\rho_0}\frac{\partial^2 p^2}{\partial t^2} = \frac{\beta}{c_p}\frac{\partial}{\partial t}\operatorname{div}\langle \mathbf{S}\rangle. \tag{1}$$

Here, p is sound pressure; c_0, ρ_0, and ε are equilibrium values of sound velocity, density, and nonlinearity of the medium; β is the coefficient of thermal expansion; c_p is specific heat; $\langle \mathbf{S}\rangle$ is the averaged Poynting vector of modulated electromagnetic waves, where averaging is performed over rapid carrier frequency oscillations.

Let the trace of a light beam move along the x axis, which lies in the boundary plane between two media (for example, the air–water interface), so that it is incident along the z normal from the transparent medium (air) on the weakly absorbing medium (water). In this case,

$$\langle \mathbf{S}\rangle_z = I_0 \exp\left(-\frac{z}{l}\right)\Phi(kx - ckt). \tag{2}$$

For simplicity, consider two-dimensional geometry of the problem; then, the dependence of all the quantities on the third y coordinate is ignored. In Eq. (2), I_0 is the characteristic intensity value at boundary $z = 0$, l is the light penetration depth in the weakly absorbing medium, Φ is the function describing light intensity distribution on the surface, and c is the beam scanning velocity. For accepted form (2), the right-hand side of Eq. (1) takes the form

$$\frac{ck\beta}{c_p l} I_0 \exp\left(-\frac{z}{l}\right)\Phi(kx - ckt). \tag{3}$$

Sources (3) moving with velocity $c \approx c_0$ are "resonant" with the wave traveling in the positive direction of the x axis. Therefore, the "right-hand" wave is much more efficiently excited, as compared to the "left-hand" one [2–4]. This allows one to seek the solution to Eq. (1) in the form [1]

$$p = p(\varsigma = x - c_0 t, \sqrt{\mu} z, \mu t), \tag{4}$$

where $\mu \ll 1$ is a small parameter. Changing to variables (4) in Eq. (1), (3) and ignoring small second-order (μ^2) and higher-order terms, we obtain

$$\frac{\partial}{\partial \varsigma}\left[\frac{\partial p}{\partial t} + \frac{\varepsilon}{c_0\rho_0} p \frac{\partial p}{\partial \varsigma} - \frac{\beta c^2 I_0}{2c_p l}\exp\left(-\frac{z}{l}\right)\Phi(k\varsigma - t\Delta)\right] = -\frac{c_0}{2}\frac{\partial^2 p}{\partial z^2}, \tag{5}$$

where $\Delta = k(c - c_0)$ is the detuning. Exactly at resonance, were the natural and forced waves travel with identical velocities, the detuning is zero.

In the particular case of a periodic distribution of moving sources along the surface

$$\frac{\beta c^2 I_0}{2c_p l}\Phi(k\varsigma - t\Delta) = \frac{1}{2}F_0 \exp(ik\varsigma - it\Delta) + c.c. \tag{6}$$

(*c.c.* denotes a complex conjugate term and i is the imaginary unit) and with neglect of the dependence on z, solution to Eq. (5) can be sought in the form of two coupled spatial harmonics:

$$p(t,\varsigma) = \frac{1}{2}A_1(t)\exp(ik\varsigma) + \frac{1}{2}A_2(t)\exp(i \times 2k\varsigma) + c.c. \tag{7}$$

For complex harmonic amplitudes, we obtain the following system of equations:

$$\frac{dA_1}{dt} + i\gamma A_1^* A_2 = F_0\exp(-it\Delta), \quad \frac{dA_2}{dt} + i\gamma A_1^2 = 0, \tag{8}$$

where $\gamma = \varepsilon k / (2c_0\rho_0)$. Model (8) is fairly universal; it may describe a set of other problems of the theory of nonlinear oscillations and waves.

In the second example under consideration, distributed sources are absent while wave excitation occurs in a plane layer (one-dimensional resonator) through oscillation of one of the walls. The wall particle velocity varies in time according to periodic law

$$u(x = 0, t) = \Phi(\omega t), \tag{9}$$

the other wall ($x = L$) is immobile. The motion of the medium in the resonator is described by homogeneous wave equation (1). In [2–4], it is shown that, for the given boundary conditions, the problem is reduced to solving functional equation

$$Q\left(\omega t + kL - \frac{\varepsilon}{c_0}kLQ\right) - Q\left(\omega t - kL + \frac{\varepsilon}{c_0}kLQ\right) = \Phi(\omega t). \tag{10}$$

In this equation, unknown function $Q(\omega t)$ describes the form of any of the two nonlinear waves traveling in opposite directions. Near the fundamental resonance ($kL = \pi + \delta, \delta \ll 1$), in the case of a harmonic wall motion law (9), the wave form can be calculated from evolution equation

$$\frac{\partial Q}{\partial t} + \frac{\delta}{\pi}\frac{\partial Q}{\partial \tau} - \frac{\varepsilon}{c_0}Q\frac{\partial Q}{\partial \tau} = \frac{\Phi_0\omega}{2\pi}\cos(\omega\tau). \tag{11}$$

Here, t is the "slow" time, which describes the oscillation establishment process, and τ is the "fast" time, which describes high-frequency oscillations [2–4]. As before, we seek the solution in the form of a sum of two coupled harmonics:

$$Q(t,\tau) = \frac{1}{2}A_1(t)\exp\left(-i\omega\tau + i\omega\frac{\delta}{\pi}t\right) + \frac{1}{2}A_2(t)\exp\left(-i \times 2\omega\tau + i \times 2\omega\frac{\delta}{\pi}t\right) + c.c., \tag{12}$$

Then, we obtain a system of equations

$$\frac{dA_1}{dt} + i\frac{\varepsilon\omega}{2c_0}A_1^* A_2 = \frac{\omega\Phi_0}{2\pi}\exp\left(-i\omega\frac{\delta}{\pi}t\right), \quad \frac{dA_2}{dt} + i\frac{\varepsilon\omega}{2c_0}A_1^2 = 0, \tag{13}$$

which differs from Eqs. (8) in notation alone.

The third example is related to a concentrated (oscillatory) system. Consider two coupled harmonic oscillators. Each of them can be represented by a point mass m_j performing oscillations along the x axis owing to a spring with stiffness k_j ($j = 1, 2$). The masses are connected with each other by a third (nonlinear) spring whose elastic restoring force depends on the difference between the mass displacements $\zeta = x_2 - x_1$ as $k\varsigma + \beta\varsigma^2$. Periodic external force $f(t)$ is applied to the first mass.

The Lagrange function of the system described above has the form

$$L = \frac{1}{2}\left[m_1\left(\frac{dx_1}{dt}\right)^2 + m_2\left(\frac{dx_2}{dt}\right)^2\right] - \frac{1}{2}\left[k_1x_1^2 + k_2x_2^2 + k(x_2 - x_1)^2\right] - \frac{\beta}{3}(x_2 - x_1)^3 + f(t)x_1. \tag{14}$$

Let us restrict our consideration to the weak coupling case, where natural frequencies $\omega_{1,2}$ of the system little differ from partial frequencies $n_{1,2}$:

$$\omega_1^2 = n_1^2 - \frac{s_1^2 s_2^2}{n_2^2 - n_1^2}, \quad \omega_2^2 = n_2^2 + \frac{s_1^2 s_2^2}{n_2^2 - n_1^2}. \tag{15}$$

Here, the following notations are used:

$$n_1^2 = \frac{k_1 + k}{m_1}, \quad n_2^2 = \frac{k_2 + k}{m_2}, \quad s_1^2 = \frac{k}{m_1}, \quad s_2^2 = \frac{k}{m_2}. \tag{16}$$

Assuming that the coupling factor is small,

$$\nu = \frac{s_1 s_2}{\left|n_2^2 - n_1^2\right|} \ll 1, \tag{17}$$

we change to the following normal variables in Eq. (14) for Lagrange function:

$$x_1 = \frac{1}{\sqrt{m_1}}(z_1 - \nu z_2), \quad x_2 = \frac{1}{\sqrt{m_2}}(\nu z_1 + z_2). \tag{18}$$

These variables reduce (accurate to ν^2) both quadratic forms appearing in Eq. (14) to diagonal form:

$$L = \frac{1}{2}\left[\left(\frac{dz_1}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2\right] - \frac{1}{2}\left(n_1 z_1^2 + n_2 z_2^2\right) - \frac{\beta}{3}\left(\frac{z_2}{\sqrt{m_2}} - \frac{z_1}{\sqrt{m_1}}\right)^3 + \frac{f(t)}{\sqrt{m_1}} z_1. \tag{19}$$

Naturally, in derivation of Eq. (19), the nonlinear term and the driving force are also assumed to be small quantities on the order of ν, Eq. (17). Equations of motion corresponding to Lagrange function (19) are as follows:

$$\begin{aligned} \frac{d^2 z_1}{dt^2} + n_1^2 z_1 &= \frac{\beta}{\sqrt{m_1}}\left(\frac{z_2}{\sqrt{m_2}} - \frac{z_1}{\sqrt{m_1}}\right)^2 + \frac{f(t)}{\sqrt{m_1}}, \\ \frac{d^2 z_2}{dt^2} + n_2^2 z_2 &= -\frac{\beta}{\sqrt{m_2}}\left(\frac{z_2}{\sqrt{m_2}} - \frac{z_1}{\sqrt{m_1}}\right)^2. \end{aligned} \tag{20}$$

The left-hand parts of Eqs. (20) are independent; owing to the change to normal coordinates, the subsystems formally interact through nonlinear coupling alone.

We assume that the first subsystem is tuned to frequency ω_0, which is approximately identical to external force oscillation frequency ω, whereas the second subsystem is tuned to frequency $2\omega_0$. In other words, the partial frequencies involved in Eqs. (20) are identical to ω_0 and $2\omega_0$, respectively. Seeking the solution to Eqs. (20) in the form

$$\begin{aligned} z_1 &= \frac{1}{2}\frac{A_1(t)}{\omega_0}\exp(-i\omega_0 t) + c.c., \\ z_2 &= \frac{1}{2}\frac{A_2(t)}{2\omega_0}\exp(-i \times 2\omega_0 t) + c.c., \\ f &= f_0 \exp(-i\omega t), \end{aligned} \tag{21}$$

where $A_{1,2}$ are slowly varying amplitudes, we arrive at a pair of reduced equations, which coincide in form with Eqs. (8). In this case, the coefficients involved in system (8) are expressed as follows:

$$\gamma = \frac{\beta}{4m_1\sqrt{m_2}\omega_0^2}, \quad F_0 = i\frac{f_0}{2\sqrt{m_1}}, \quad \Delta = \omega - \omega_0. \tag{22}$$

Note that functions $A_{1,2}$ describe velocity oscillations rather than displacement ones. Dividing these functions by the corresponding frequency values in Eq. (21), we achieve equality of nonlinear coefficients denoted by γ in system (8), which leads to the evident energy relation:

$$\frac{d}{dt}\left[\left|A_1\right|^2 + \left|A_2\right|^2\right] = 2\,\mathrm{Re}\left[A_1^* F_0 \exp(-it\Delta)\right]. \tag{23}$$

In the absence of external action, from Eq. (23) it follows that the total energy of first and second harmonic oscillations is conserved. For displacement oscillation amplitudes, such a relation does not hold.

It should be added that the problems discussed above only illustrate the possibilities of using model equations (8) in acoustics. Evidently, there exists a great number of oscillatory and wave processes of other physical nature (e.g., in nonlinear optics, electrodynamics of intense short-wave radiation sources, and many other fields of physics) that can also be described by a pair of "reduced" equations for first and second harmonic amplitudes [5] in the presence of an external "pumping" source.

ANALYSIS OF EQUATIONS WITH ALLOWANCE FOR SELECTIVE ABSORPTION

Let us consider model (8) with its second equation including term αA_2 responsible for second harmonic absorption:

$$\begin{aligned} \frac{dA_1}{dt} + i\gamma A_1^* A_2 &= F_0 \exp(-it\Delta), \\ \frac{dA_2}{dt} + \alpha A_2 + i\gamma A_1^2 &= 0. \end{aligned} \tag{24}$$

Ways of technical implementation of such an absorption depend on the specific system structure, and, hence, consideration of the problem in general form is of no use. Some examples related to acoustics can be found in [6–8].

Mathematical model (24) is most conveniently analyzed using dimensionless variables

$$\begin{aligned} A_1 &= \left(\frac{\alpha F_0}{\gamma^2}\right)^{1/3} \exp(-it\Delta) B_1, \\ A_2 &= \left(\frac{F_0^2}{\alpha\gamma}\right)^{1/3} \exp(-it \times 2\Delta) B_2. \end{aligned} \tag{25}$$

Introduction of Eqs. (25) in Eqs. (24) yields

$$\begin{aligned} t_1\left(\frac{dB_1}{dt} - i\Delta B_1\right) + iB_1^* B_2 &= 1, \\ t_2\left(\frac{dB_2}{dt} - i \times 2\Delta B_2\right) + B_2 + iB_1^2 &= 0. \end{aligned} \tag{26}$$

In the system of equations (26), we choose three constant quantities with dimension of time:

$$t_1 = \left(\frac{\alpha}{\gamma^2 F_0^2}\right)^{1/3}, \quad t_2 = \alpha^{-1}, \quad t_3 = \Delta^{-1}. \tag{27}$$

Since Eqs. (26) cannot be solved in general form, let us analyze them for several limiting cases differing in relative values of characteristic times (27).

The problem is stated as follows: to solve Eqs. (26) with zero initial conditions

$$B_1(t = 0) = 0, \quad B_2(t = 0) = 0. \tag{28}$$

This means that, at the initial instant of time, an external source pumping energy to the first harmonic is tuned on. Nonlinearity leads to partial energy transfer to the double-frequency oscillation. The increase in both harmonic amplitudes is determined not only by harmonic coupling but also by selective loss. In this situation, the main problem is related to the effect of loss on the dynamic balance of oscillation energy.

Equations (26) are written for complex amplitudes. Subsequent consideration will also need equations for real amplitudes b_1, b_2 and phases s_1, s_2 of interacting harmonics. From Eqs. (26), with the use of substitution

$$B_1 = b_1 \exp(is_1), \quad B_2 = b_2 \exp(is_2), \tag{29}$$

we obtain

$$t_1 \frac{db_1}{dt} - b_1 b_2 \sin(s_2 - 2s_1) = \cos(s_1), \tag{30}$$

$$t_1 b_1 \left(\frac{ds_1}{dt} - \Delta\right) + b_1 b_2 \cos(s_2 - 2s_1) = -\sin(s_1), \tag{31}$$

$$t_2 \frac{db_2}{dt} + b_2 + b_1^2 \sin(s_2 - 2s_1) = 0, \tag{32}$$

$$t_2 b_2 \left(\frac{ds_2}{dt} - 2\Delta\right) + b_1^2 \cos(s_2 - 2s_1) = 0. \tag{33}$$

First, let us consider steady-state oscillations by assuming that the time derivatives of amplitudes and phases involved in Eqs. (29)–(33) are zero. In this case, the system of four algebraic equations is reduced to a cubic equation in the first harmonic amplitude squared:

$$b_1^6 - 4\Delta^2 t_1 t_2 b_1^4 + (\Delta t_1)^2 \left[1 + (2\Delta t_2)^2\right] b_1^2 - \left[1 + (2\Delta t_2)^2\right] = 0. \tag{34}$$

After solving Eq. (34), it is possible to calculate the second harmonic amplitude by the formula

$$b_2^2 = b_1^4 \left[1 + (2\Delta t_2)^2\right]^{-1}. \tag{35}$$

In the simplest case of zero detuning, Eqs. (34), (35) yield $b_1^2 = b_2^2 = 1$. Returning to dimensional variables through Eqs. (25), for steady-state amplitude values we obtain

$$|A_1| = \left(\frac{\alpha F_0}{\gamma^2}\right)^{1/3}, \quad |A_2| = \left(\frac{F_0^2}{\alpha\gamma}\right)^{1/3}. \tag{36}$$

One can easily show that solution (36) is stable.

Result (36) demonstrates a tendency that is unexpected at first glance: as the second harmonic absorption coefficient α increases, the fundamental frequency oscillation amplitude grows. Such a relation was mentioned in, e.g., [9, 10], but the authors of the cited publications considered nondispersive nonlinear waves characterized by shock front formation and energy "spread" over higher harmonics. For such waves, second harmonic suppression "blocks" the cascade process of energy transfer toward upper part of the spectrum and "turns off" nonlinear absorption. This problem was discussed earlier in [11, 12]. Now, it appears that an increase in energy retained at fundamental frequency with increasing nonlinear loss takes place even in simple nonlinear systems (24) with only two interacting harmonics [13, 14].

SUPPRESSION OF NONLINEAR DISTORTIONS AND AN INCREASE IN THE Q FACTOR OF A RESONATOR WITH SELECTIVE LOSS

As shown above, the effect of selective absorption on the nonlinear interaction process is of general significance from the point of view of oscillation and wave theory. By introducing selectively absorbing elements in a system (medium), it is possible to control the energy fluxes. One of phenomena of interest is as follows. Setting up partial energy transfer from an acoustic cavity by introducing selective dissipation at second harmonic frequency, it is possible to considerably increase nonlinear oscillations and raise the stored energy and the Q factor. The paradoxical effect of Q factor increase clearly manifests itself in the cases where higher harmonic frequencies generated in the nonlinear medium are close to the resonator frequencies. An important example of a system with corresponding properties is an acoustic resonator with selective loss.

The eigenfrequency spectrum of a resonator with stiff walls is equidistant; i.e., $\omega_n = n\omega_0 = n\pi c/L$. Therefore, the generated harmonic with number n is the nth mode. Therefore, a cascade of nonlinear processes causing efficient energy transfer to the upper part of the spectrum occurs in the resonator. In the high-frequency region, oscillation energy is strongly absorbed because of dissipative processes, which are usually related to viscosity and thermal conductivity of the medium.

General ideas of wave interaction control by introduction of selective losses are described in [9, 14–16].

In the case under consideration, it is necessary to introduce an absorber for frequency $2\omega_0$; the second harmonic suppression stops the cascade process of energy transfer to the upper part of the spectrum or, in other words, "suppresses" shock front formation. Technically, the loss at frequency $2\omega_0$ can be implemented by either introducing resonant scattering elements in the medium (e.g., gas bubbles in liquid) or using selective boundaries (e.g., transparent for $2\omega_0$ waves and reflecting all other frequencies [6–8, 16]).

Let us represent the field oscillating between resonator walls $x = 0$ and $x = L$ as superposition of two nonlinear waves propagating in opposite directions [2, 4]. Function u describing the "right-hand" particle velocity wave obeys the equation

$$\frac{1}{c}\frac{\partial u}{\partial t} - \frac{\varepsilon}{c^2}u\frac{\partial u}{\partial \tau} - \frac{b}{2c^3\rho}\frac{\partial^2 u}{\partial \tau^2} = \frac{A}{2L}\sin\omega t - \frac{\alpha}{c}b_2(t)\sin 2\omega t. \quad (37)$$

Here, t is the "slow" time describing the setting processes in the resonator; τ is the "fast" time describing oscillations; α is the selective absorption factor; and $b_2(t)$ is the second harmonic amplitude

$$b_2(t) = \frac{2}{\pi}\int_0^\pi u(t,\tau)\sin 2\omega\tau d(\omega\tau), \quad (38)$$

which is preliminarily unknown. Thus, model (37), (38) has the form of an integro-differential equation. In the cases where the right-hand side of Eq. (37) is known, the model takes the form of an inhomogeneous Burgers equation.

Let us introduce dimensionless variables

$$V = \frac{u}{u_0},\quad \theta = \omega\tau,\quad T = \frac{t}{t_{SH}};$$
$$t_{SH} = \frac{c}{\varepsilon\omega u_0},\quad u_0 = \sqrt{\frac{Ac}{2\pi\varepsilon}}. \quad (39)$$

Here, t_{SH} is the characteristic "nonlinear" time within which a wave may develop a discontinuity and u_0 is the characteristic amplitude. In terms of notation (39), the model takes the form

$$\frac{\partial V}{\partial T} - V\frac{\partial V}{\partial \theta} - \Gamma\frac{\partial^2 V}{\partial \theta^2} = \sin\theta - D\sin 2\theta\frac{2}{\pi}\int_0^\pi V(T,\theta')\sin 2\theta' d\theta'. \quad (40)$$

Here, the following dimensionless quantities are present:

$$\Gamma = \frac{b\omega}{2\varepsilon\rho c u_0} = \frac{t_{SH}}{t_{DIS}},\quad D = \frac{\alpha c}{\varepsilon\omega u_0} = \alpha t_{SH}. \quad (41)$$

They are determined by the ratio of nonlinear time t_{SH} to the time of conventional dissipative (viscous) absorption (this refers to quantity Γ) or to characteristic time α^{-1} of selective absorption (quantity D).

To calculate the forced oscillation excitation process in the resonator, it is necessary to solve Eq. (40) with zero initial condition $V(T = 0, \theta) = 0$. For large "slow" time values $T \to \infty$, balance is achieved between the energy supplied by the source (the oscillating wall) and the three types of loss: viscous, nonlinear, and selective. Analysis is only possible for the case of steady-state oscillations. Under steady-state conditions, nonlinearity manifests itself most strongly and, hence, this case is of most interest.

Let us perform integration by assuming (for simplicity) that viscosity is zero:

$$V^2 - \overline{V^2} = 2\cos\theta - DB_2\cos 2\theta. \quad (42)$$

Here, the overbar indicates averaging over the oscillation period. Constant quantity $\overline{V^2}$ is proportional to the mean wave intensity. We determine it as $\overline{V^2} = 2 + DB_2$ using condition $V(\theta = \pi) = 0$. Then, the solution has the form

$$\frac{V_{ST}(\theta)}{\sqrt{2}} = \pm\left[(1+\cos\theta) + D(1-\cos^2\theta)\frac{2}{\pi}\times\int_0^\pi V_{ST}(\theta')\sin 2\theta' d\theta'\right]^{1/2}. \quad (43)$$

In Eq. (43), the plus sign is taken for half-period $0 < \theta \le \pi$ and the minus sign for $-\pi \le \theta < 0$. In the vicinity of point $\theta = 0$, a shock front is formed. Without considering its structure, we set $\Gamma = 0$. Finiteness of Γ can be taken into account by the method of matched asymptotic expansions, which yields small corrections (under strong nonlinearity conditions) to energy characteristics of the field.

Figure 1 shows the profiles of one oscillation period (43) for different values of selective absorption factor $D = 0, 1, 4, 10, 20$. In the presence of nonlinear absorption alone ($D = 0$), the profile corresponds to the well-known solution to nonlinear Burgers equation [14, 15]:

$$V_{ST}(\theta) = 2\cos(\theta/2)\,\mathrm{sign}\,\theta. \quad (44)$$

According to Fig. 1, as selective absorption D increases, the dimensionless discontinuity amplitude does not grow but an increase is observed in disturbance V_{ST} within smooth parts of the profile. For $D \gg 1$, oscillation is described by expression $V_{ST} \approx V_0\sin\theta$. It is almost harmonic, but, at point $\theta = 0$, a discontinuity

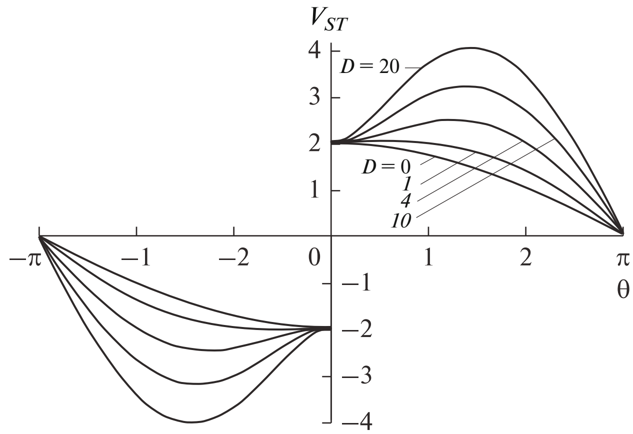

Fig. 1. One-period profiles of one of the two traveling waves forming the nonlinear field in a resonator with selective loss D at the second harmonic frequency: $D = 0, 1, 4, 10, 20$.

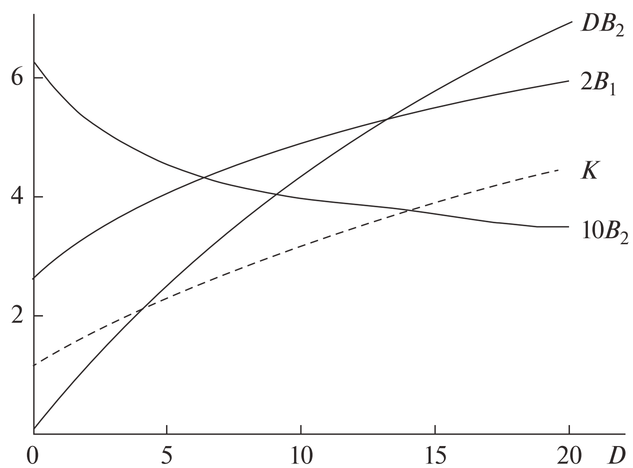

Fig. 2. First and second harmonic amplitudes and product DB_2 versus selective absorption D (solid lines); "amplification factor" K of oscillation energy accumulated in the cavity (dashed line).

persists. However, for large values of D, the dimensionless value of the jump (=2) is small, as compared to wave amplitude V_0; i.e., $2 \ll V_0$.

Thus, as D increases, second harmonic amplitude B_2 corresponding to frequency 2ω considerably decreases. The onset of this process is shown in Fig. 2. Suppression of 2ω wave decelerates the energy transfer to higher harmonics 3ω, $4\omega, \ldots,$ and, therefore, energy is accumulated in the fundamental frequency wave, ω, which is almost undamped. An increase in first harmonic amplitude $B_1(D)$ is also shown in Fig. 2. In addition, Fig. 2 shows dependence $DB_2(D)$. The maximum value of disturbance (43) $V_{ST} = 2$ is reached at $\theta = 0$ for $DB_2 \le 0.5$; for $DB_2 > 0.5$, the maximum (see Fig. 1) is shifted to point $\theta_{\max}$, where

$$V_{\max}(\theta_{\max}) = \frac{1 + 2DB_2}{\sqrt{2DB_2}},\quad \theta_{\max} = \arccos\frac{1}{2DB_2}. \tag{45}$$

The mean wave intensity also increases with increasing selective absorption:

$$I = \overline{V_{ST}^2} = 2 + DB_2(D). \tag{46}$$

In the case of nonlinear oscillations with complex spectral composition, the Q factor of the resonator can be determined as the ratio of maximum velocity perturbation in the standing wave, $2u_{\max}$, to velocity amplitude A of boundary oscillations:

$$Q = \frac{2u_{\max}}{A} = \sqrt{\frac{2c}{\pi\varepsilon A}}\Phi(2DB_2);\quad \Phi(x \equiv 2DB_2) = \frac{1 + x}{\sqrt{x}},\quad x > 1;\quad \Phi = 2,\quad x \le 1. \tag{47}$$

It is also possible to determine the square of the Q factor through the mean intensity ratio of these oscillations:

$$Q^2 = \frac{2\overline{u^2}}{A^2/2} = \frac{2c}{\pi\varepsilon A}(2 + DB_2). \tag{48}$$

Both formulas (47) and (48) describe the increase in the Q factor with increasing selective absorption D.

Estimates obtained from these formulas show that, if the right-hand wall $x = L$ of the resonator selectively transmits 98% of incident radiation power at the second harmonic frequency to outer space, the Q factor of the resonator increases by a factor of about 3.5 while oscillation energy increases by an order of magnitude.

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